Hensel-lifting black-box algorithms and fast trace computation for elliptic-curve endomorphisms

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Motivation: computing traces

Main characters:

- Elliptic curves over a finite field $\mathbb{F}_q$, $\quad q = p^m$.
- An endomorphism $\varphi \in \mathrm{End}(E)$, over $\mathbb{F}_q$ as well.

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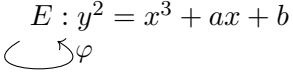

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$$\varphi^2 - [t]\varphi + [d] = 0, \qquad t = \mathrm{tr}(\varphi), \quad d = \deg(\varphi).$$

- The degree $d = \deg \varphi$ is usually intrinsic to the representation of φ on a computer.
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Cost of computing $\mathrm{tr}\,\varphi$ on a curve $E/\mathbb{F}_{p^m}$, where φ is stored via $n = O(\log pd)$ field elements

[BCEMP18] `arXiv 1804.04063`	$\widetilde{O}(n^7)$	bit ops
[MPSW25] `arXiv 2501.16321`	$O(n^4 \log(n)^2)$	bit ops
this work	$\widetilde{O}(n^2)$	bit ops

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- Satisfies quadratic equation
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- Satoh's algorithm (Satoh 2000, ...): different approach $\nleftrightsquigarrow$ we adapt this one!
  - p-adic lifting of curves and endomorphisms
  - action of morphisms on invariant differentials

Ingredient #1: p-adic lifting, algorithmically

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$$\mathbb{Z}_q = W(\mathbb{F}_q) \;\approx\; \{\sum_{i=0}^{\infty} a_i\, p^i \mid a_i \in \mathbb{F}_q\} \qquad \sum a_i p^i \mapsto a_0 \in \mathbb{F}_q$$

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$$F(a + p^k v) = F(a) + p^k \mathrm{D}F(a) \cdot v + O(p^{2k}) \overset{?}{=} 0 \pmod{p^{2k}}$$

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No need to know $F, \mathrm{D}F$ explicitly! ✓ a black-box algo $a \mapsto F(a)$ is sufficient:

$$\begin{array}{rcll} a & \mapsto & F(a) & + O(p^{2k}) \\ a + e_i p^k & \mapsto & F(a) \;+ \partial_i F(a) p^k & + O(p^{2k}) \end{array}$$

$n+1$ black-box calls
$\rightsquigarrow$ get $F(a), \mathrm{D}F(a)$

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- Lifting φ = ? $\quad$ ... depends on the algorithmic representation of φ.

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$$E = E_0 \xrightarrow{\varphi_1} E_1 \xrightarrow{\varphi_2} E_2 \xrightarrow{\varphi_3} \dots \xrightarrow{\varphi_r} E_n = E', \qquad \varphi : E \to E' \text{ (over the arc)}$$

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$\exists\ 2^e$-isogeny of principally polarized abelian surfaces $\Phi\colon E \times E_a \to E' \times E_b$ s.t.

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$\rightsquigarrow$ we get a *smooth-degree* isogeny chain (in dim 2), easy to lift
$\rightsquigarrow$ + quirky algebraic ($\implies$ liftable!) constraints coming from the 2-dim embedding.

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where as a rational map, φ looks like:

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Lifting preserves algebraic relations: $\mathrm{tr}(\widetilde{\varphi}) = \mathrm{tr}(\varphi)$.
$\rightsquigarrow$ Now $\mathrm{tr}(\widetilde{\varphi})$ lies in $\mathbb{Z} \hookrightarrow R$, i.e., an actual integer!

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Bulk of the computation: lifting to the highest precision.
i.e., $\approx$ re-evaluating the isogeny a handful of times at precision k.

Complexity (in bit operations) when φ isogeny chain:
(# steps) $\times$ (cost of a step) $\times$ (cost of multiplications in highest precision: $\mathbb{Z}_q/p^k\mathbb{Z}_q$) $=$

$$n \cdot O(\ell) \cdot \widetilde{O}(k) = \widetilde{O}(n^2), \qquad k \approx \log d \approx n, \quad \ell = O(1)$$

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Thank you for your attention :D

Questions?

Bonus details: Lifting a single isogeny step

Let $E_i : y^2 = x^3 + a_i x + b_i$ over $\mathbb{F}_q$ for $i = 0, 1$, $\varphi : E_0 \to E_1$ an ℓ-isogeny, ℓ small.

First lift the domain curve to $\widetilde{E_0} : y^2 = x^3 + \widetilde{a_0} x + \widetilde{b_0}$.

(sqrt)Vélu: Say $\ker \varphi = \langle P \rangle$. How do we lift to $\widetilde{P}$?

- $\widetilde{P}$ is of order ℓ: lift root of division polynomial $\psi_{\widetilde{E},\ell}(x)$, get $x_P \rightsquigarrow \widetilde{x_P}$

  
  better: $\psi_{\widetilde{E},\ell}$ depends polynomially on the curve: $\psi_\ell(a, b, x)$.
  $\rightsquigarrow$ compute partial derivatives wrt a, b, x $\rightsquigarrow$ linear dependency between $\widetilde{a_0}, \widetilde{b_0}, \widetilde{x_P}$

- $\widetilde{P}$ lies on $\widetilde{E}$: lift root of defining polynomial $y^2 - x^3 - \widetilde{a}x - \widetilde{b}$
  $\rightsquigarrow$ linear dependency between $\widetilde{a}, \widetilde{b}, \widetilde{x_P}, \widetilde{y_P}$

j-invariants: if we've got $(E_0, j(E_1))$, instead of a kernel point?

$\rightsquigarrow$ BMSS, 2008, computes E_1 with the correct j, and $\varphi : E_0 \to E_1$.

✓ The computation is algebraic: rational fn of the input

⚠ problems when $j(E_0)$ or $j(E_1)$ are in $\{0, 1728\}$... as usual

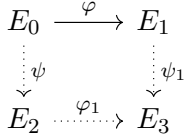
- φ non unique
- non-smooth map $j(E) \to E$

Note both this algo and Vélu compute a normalized φ (i.e. scaling factor = 1)

Lifting higher-dimensional representations

Embedding Lemma Let $\varphi\colon E_0 \to E_1$ be separable of odd degree d.
One can build a 2D polarized 2^e-isogeny $\Phi\colon E_0 \times E_3 \to E_1 \times E_2$ s.t.

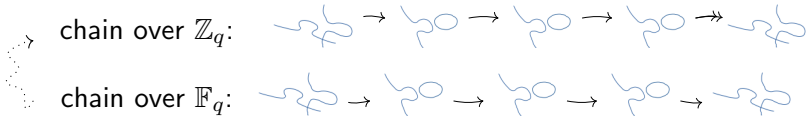
$$\varphi \quad \text{equals the composition} \quad E_0 \hookrightarrow E_0 \times E_3 \xrightarrow{\Phi} E_1 \times E_2 \twoheadrightarrow E_1.$$



The tuple (E_0, E_3, P, Q) gives an efficient representation for φ,
where $\langle P, Q\rangle = \ker \Phi \subseteq (E_0 \times E_3)[2^e]$. To lift the 2D-rep, make sure:

- both P and Q lie on the domain product surface and have order 2^e
- the 2D isogeny with kernel P, Q lands on a product
- (extra constraint if we're lifting an endo): $j(E_0) = j(E_1)$

} all algebraic



Same strategy generalizes to 4D, 8D representations.