Rational points on modular curves via maps to elliptic curves with rank zero

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Code available at `github.com/rouseja/ModCrvToEC`

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- Given $\sigma \in G_K := \mathrm{Gal}(\overline{K}/K)$, σ acts as an automorphism on $E[n]$.
- This gives a homomorphism $\rho_{E,n} \colon G_K \to \mathrm{Aut}(E[n]) \simeq \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$.

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$$P = (11 + 5\sqrt{11}, \frac{5}{2}(1 - i)(11 + 3\sqrt{11})\sqrt[4]{-11})$$

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- $\mathrm{Gal}(\mathbb{Q}(E[4])/\mathbb{Q}) = \langle \sigma, \tau \rangle$ where

$$\sigma(i) = i \text{ and } \sigma(\sqrt[4]{-11}) = i\sqrt[4]{-11}$$
$$\tau(i) = -i \text{ and } \tau(\sqrt[4]{-11}) = \sqrt[4]{-11}.$$

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- We have $\sigma(P) = 3P + 2Q$ and $\sigma(Q) = P + Q$, and so

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- Thus, $\operatorname{im} \rho_{E,4} = \left\langle \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \right\rangle \leq \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$.

Why study Galois representations?

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• Galois representations naturally arise in the study of class numbers of imaginary quadratic fields, Fermat's Last Theorem, and the inverse Galois problem.

• If E/K is an elliptic curve, one can consider the single representation

$$\rho_E \colon G_K \to \varprojlim \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}) := \mathrm{GL}_2(\hat{\mathbb{Z}}).$$

Mazur's Program B

B. Given a number field K and a subgroup H of $GL_2\widehat{\mathbb{Z}} = \prod_p GL_2\mathbb{Z}_p$ classify all elliptic curves $E_{/K}$ whose associated Galois representation on torsion points maps $\mathrm{Gal}(\overline{K}/K)$ into $H \subset GL_2\widehat{\mathbb{Z}}$.

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- We say that E has complex multiplication (CM) if $\mathrm{End}_{\overline{K}}(E)$ is an order in an imaginary quadratic field.

Modular curves

• If H is an open subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$, there is a curve X_H together with a map $\pi_H \colon X_H \to \mathbb{P}^1$ with the property that if E/K is an elliptic curve,

$$\mathrm{im}\ \rho_E \subseteq \text{ a conjugate of } H \iff j(E) \in \pi_H(X_H(K))$$

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• If $\det : H \to \hat{\mathbb{Z}}^\times$ is surjective, then X_H is a nice curve defined over $\mathbb{Q}$.

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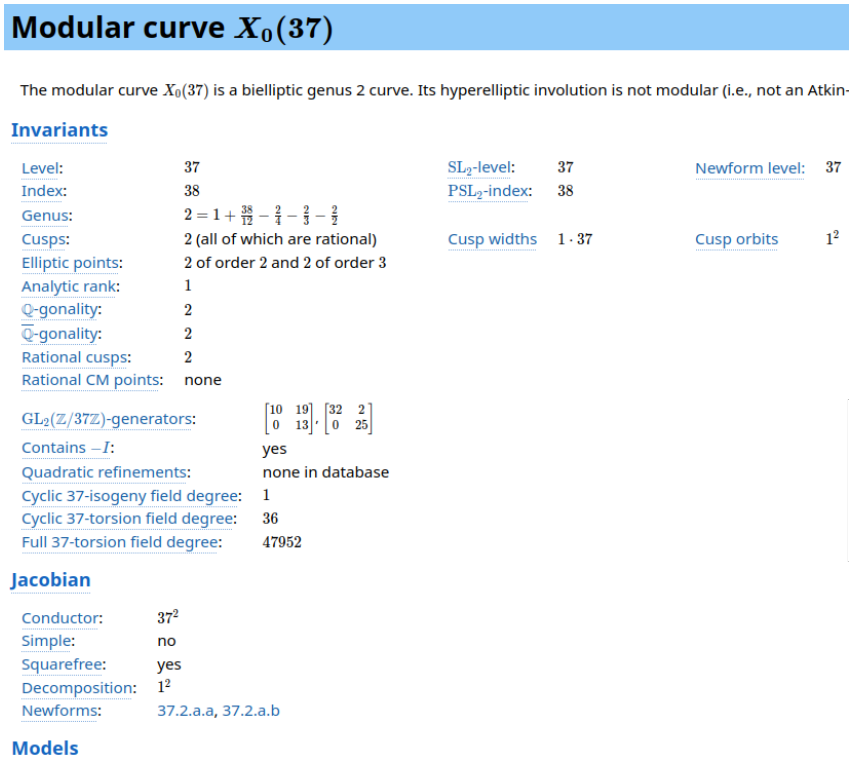

Modular curve $X_0(37)$

The modular curve $X_0(37)$ is a bielliptic genus 2 curve. Its hyperelliptic involution is not modular (i.e., not an Atkin-

Invariants

Level:	37	SL_2-level:	37	Newform level:	37
Index:	38	PSL_2-index:	38		
Genus:	$2 = 1 + \frac{38}{12} - \frac{2}{4} - \frac{2}{3} - \frac{2}{2}$				
Cusps:	2 (all of which are rational)	Cusp widths	$1 \cdot 37$	Cusp orbits	1^2
Elliptic points:	2 of order 2 and 2 of order 3				
Analytic rank:	1				
$\mathbb{Q}$-gonality:	2				
$\overline{\mathbb{Q}}$-gonality:	2				
Rational cusps:	2				
Rational CM points:	none				

$\mathrm{GL}_2(\mathbb{Z}/37\mathbb{Z})$-generators:	$\begin{bmatrix}10 & 19\\ 0 & 13\end{bmatrix}, \begin{bmatrix}32 & 2\\ 0 & 25\end{bmatrix}$
Contains $-I$:	yes
Quadratic refinements:	none in database
Cyclic 37-isogeny field degree:	1
Cyclic 37-torsion field degree:	36
Full 37-torsion field degree:	47952

Jacobian

Conductor:	37^2
Simple:	no
Squarefree:	yes
Decomposition:	1^2
Newforms:	37.2.a.a, 37.2.a.b

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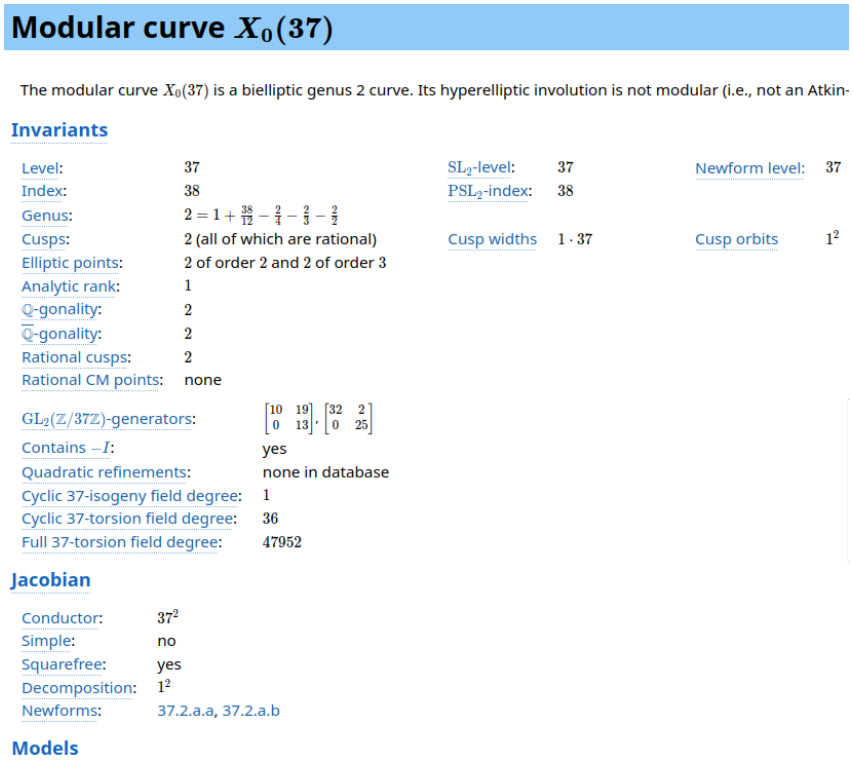

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Models

• The database includes data about labels, subgroups, rational points, equations for X_H, the map $\pi_H \colon X_H \to \mathbb{P}^1$, and the decomposition of the Jacobian $J(X_H)$.

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• So if we can determine ϕ explicitly, we can use it to determine $X_H(\mathbb{Q})$.

Main result

• There are 804450 open subgroups $H \leq \mathrm{GL}_2(\hat{\mathbb{Z}})$ with level ≤ 70 that contain $-I$. For 797591 of these, there is a map $X_H \to E$ for some elliptic curve $E/\mathbb{Q}$ with rank zero.

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Theorem (Mayle-R, 2026)

We determine $\pi_H(X_H(\mathbb{Q}))$ for all but 10 of the 797591. The ten cases we are unable to handle all have genus > 60.

Minimal modular curves

- If $H \leq G$ are open subgroups of $\mathrm{GL}_2(\hat{\mathbb{Z}})$, there is a natural map $X_H \to X_G$ and as a consequence, $J(X_G)$ is an isogeny factor of $J(X_H)$.

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• We are able to handle X_H with genus as large as 54. In some cases the degree of $X_H \to E$ is as large as 126.

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- We compute the period lattice

$$\Lambda = \left\{ \int_z^{g(z)} f(s)\, ds : g \in H \cap \mathrm{SL}_2(\mathbb{Z}) \right\}.$$

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$$X_H(\mathbb{C}) \to \mathbb{H}^*/(H \cap \mathrm{SL}_2(\mathbb{Z})) \to \mathbb{C}/\Lambda \to E(\mathbb{C}),$$

where the map from $\mathbb{H}^*/(H \cap \mathrm{SL}_2(\mathbb{Z})) \to \mathbb{C}/\Lambda$ is given by

$$P \mapsto \int_{\alpha(Q)}^{\alpha(P)} f(s)\, ds.$$

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7. Compute degrees of the maps to the elliptic curves, and pick the elliptic curve E' and modular form f for which the degree is minimal.

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14. Compute $j(P)$ for $P \in X_H(\mathbb{Q})$.

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• For good primes $\ell > p$,

$$|Z(\mathbb{Q})| \leq |Z(\mathbb{Q}_\ell)| = |Z(\mathbb{F}_\ell)|.$$

Search for ℓ where the upper bound matches our lower bound.

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• In one genus 11 case, the periods computed did not generate the period lattice. In one genus 50 case, the computation in Step 13 failed.

Bonus: a map to an elliptic curve over $\mathbb{Q}(\sqrt{-3})$

- Let X_H be the modular curve parametrizing elliptic curves whose mod 3 image of Galois is contained in the normalizer of the non-split Cartan and mod 7 image of Galois is contained in the normalizer of the split Cartan.

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• We have $J(X_H) \sim E \times A$, where E is a rank one elliptic curve, and A is a rank zero abelian surface that factors as the square of an elliptic curve E' over $\mathbb{Q}(\sqrt{-3})$.

Bonus: a map to an elliptic curve over $\mathbb{Q}(\sqrt{-3})$

• Let X_H be the modular curve parametrizing elliptic curves whose mod 3 image of Galois is contained in the normalizer of the non-split Cartan and mod 7 image of Galois is contained in the normalizer of the split Cartan.

• We have $J(X_H) \sim E \times A$, where E is a rank one elliptic curve, and A is a rank zero abelian surface that factors as the square of an elliptic curve E' over $\mathbb{Q}(\sqrt{-3})$.

• We find a degree 6 map $\phi : X_H \to E'$ defined over $\mathbb{Q}(\sqrt{-3})$ and use this to determine $X_H(\mathbb{Q}(\sqrt{-3}))$.

Summary

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• We compute these maps and use this to determine $j(X_H(\mathbb{Q}))$ for all but 10 such modular curves.

Thank you!