Affine Chabauty II

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slides: `https://sites.google.com/view/mariusleonhardt/`

code: `https://github.com/martinluedtke/AffChab2`

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Affine Chabauty I

Setup

- $X/\mathbb{Q}$ smooth projective curve
- $D \subset X$ nonempty ("cusps")
- $Y = X \setminus D$ affine curve
- S finite set of primes
- $\mathcal{X}/\mathbb{Z}$ regular model of X
- $\mathcal{D}$ the closure of D in $\mathcal{X}$
- $\mathcal{Y} = \mathcal{X} \setminus \mathcal{D}$ model of Y

- $D = \{\pm\infty\}$
- $Y \colon y^2 = d^2 x^{2g+2} + \dots$
- $S = \emptyset$

Main goal: Compute $\mathcal{Y}(\mathbb{Z}_S)$.

e.g. $$Y \colon y^2 = x^6 + 2x^5 - 7x^4 - 18x^3 + 2x^2 + 20x + 9$$

Affine Chabauty I

- g genus of X
- $r = \mathrm{rk}\, J(\mathbb{Q})$
- $n = \#D(\overline{\mathbb{Q}}) = n_1 + 2n_2$
- $p \notin S$ and base point $P_0 \in Y(\mathbb{Q})$

Partition $\mathcal{Y}(\mathbb{Z}_S) = \bigsqcup_\Sigma \mathcal{Y}(\mathbb{Z}_S)_\Sigma$ by reduction type $\Sigma = (\Sigma_\ell)_\ell$.

Theorem (L.-Lüdtke, 2025 + 2026)

If $\boxed{r + \#S + n_1 + n_2 - \#|D| < g + n - 1}$*, then there exist a computable log differential* ω *and* $c \in \mathbb{Q}_p$ *s.t.*

$$\mathcal{Y}(\mathbb{Z}_S)_\Sigma \subset \left\{ P \in \mathcal{Y}(\mathbb{Z}_p) \;\middle|\; \int_{P_0}^{P} \omega = c \right\} =: \mathcal{Y}(\mathbb{Z}_p)_{\omega,c}.$$

Affine Chabauty I: remarks

Theorem (L.-Lüdtke, 2025)

If $r + \#S + n_1 + n_2 - \#|D| < g + n - 1$, *then* $\exists(\omega, c)$ *s.t.*

$$\mathcal{Y}(\mathbb{Z}_S)_\Sigma \subset \left\{ P \in \mathcal{Y}(\mathbb{Z}_p) \;\middle|\; \int_{P_0}^{P} \omega = c \right\}.$$

Remark

- Chabauty $r < g \rightsquigarrow$ now $r + \#S < g + n - 1$ if $D(\mathbb{Q}) = D(\overline{\mathbb{Q}})$.
- Corollary: good bound on $\#\mathcal{Y}(\mathbb{Z}_S)$.
- Have a version for curves over number fields.
- Works for bad p and nonempty S.
- No restriction on type of curve (e.g. hyper-/superelliptic).

Examples

Split even degree hyperelliptic curves, $r = g$

- 6081.b.164187.1, $D = \{\infty_\pm\}$: $\mathbb{Z}$-points on $Y\colon y^2 = x^6 + 2x^5 - 7x^4 - 18x^3 + 2x^2 + 20x + 9$ are
$$\{(-1, \pm 1),\ (0, \pm 3),\ (1, \pm 3),\ (-2, \pm 3),\ (-4, \pm 37)\}.$$
- 1549.a.1549.1, $D = \{\infty_\pm\}$: $\mathbb{Z}[\zeta_3]$-points on $Y\colon y^2 = x^6 - 4x^5 + 2x^4 + 6x^3 + x^2 - 10x + 1$ are
$$\left\{(0, \pm 1),\ (2, \pm 1),\ (1, \pm\sqrt{-3})\right\}.$$

Superelliptic curves, $r = g = 1$

$Y_a\colon y^3 = x^3 + ax^2 + x$ with $a \in \mathbb{Z} \setminus \{\pm 2\}$, $D = \{(1 : \zeta_3^i : 0) | i = 0, 1, 2\}$

- $a = 1$: $\mathbb{Z}[\frac{1}{487}]$-points* on $y^3 = x^3 + x^2 + x$ are contained in $\{(0,0), (\zeta_3, 0), (\zeta_3^{-1}, 0), (\frac{216}{487}, \frac{438}{487}), (\frac{1}{18}, \frac{7}{18}), \left(\frac{-2-\sqrt{-3}}{7}, \frac{3+5\sqrt{-3}}{14}\right)\} \cup$ $\{6 \text{ extra } \mathbb{Z}_7\text{-points}\}$

Affine Chabauty II: algorithm

Affine Chabauty: main ideas

Theorem (L.-Lüdtke, 2025)

If $r + \#S + n_1 + n_2 - \#|D| < g + n - 1$, *then* $\exists(\omega, c)$ *s.t.*

$$\mathcal{Y}(\mathbb{Z}_S)_\Sigma \subset \left\{ P \in \mathcal{Y}(\mathbb{Z}_p) \;\middle|\; \int_{P_0}^{P} \omega = c \right\}.$$

Main ideas:

- Embed Y into the generalised Jacobian J_Y.
- Bound AJ-image of $\mathcal{Y}(\mathbb{Z}_S)$ inside $J_Y(\mathbb{Q})$ via arithmetic intersection theory:

$$\mathrm{AJ}_{P_0}(\mathcal{Y}(\mathbb{Z}_S)) \subset \bigcup_{\Sigma} \mathrm{Sel}(P_0, \Sigma).$$

- Each $\mathrm{Sel}(P_0, \Sigma)$ is a translate of a subgroup of rank $\leq r + \#S + n_1 + n_2 - \#|D|$.

Affine Chabauty diagram

Theorem (L.-Lüdtke, 2025)

If $r + \#S + n_1 + n_2 - \#|D| < g + n - 1$*, then* $\exists(\omega, c)$ *s.t.*

$$\mathcal{Y}(\mathbb{Z}_S)_\Sigma \subset \left\{ P \in \mathcal{Y}(\mathbb{Z}_p) \;\middle|\; \int_{P_0}^P \omega = c \right\}.$$

$$\begin{tikzcd}
\mathcal{Y}(\mathbb{Z}_S)_\Sigma \arrow[r, hook] \arrow[d, "\mathrm{AJ}_{P_0}"] & \mathcal{Y}(\mathbb{Z}_S) \arrow[r, hook] \arrow[d, "\mathrm{AJ}_{P_0}"] & \mathcal{Y}(\mathbb{Z}_p) \arrow[d, "\mathrm{AJ}_{P_0}"] \arrow[dr, "\int_{P_0}"] & \\
\mathrm{Sel}(P_0, \Sigma) \arrow[r, hook] & J_Y(\mathbb{Q}) \arrow[r, hook] & J_Y(\mathbb{Q}_p) \arrow[r, "\log_{J_Y}"] & \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D))^\vee
\end{tikzcd}$$

$\dim \mathrm{LHS} \leq r + \#S + n_1 + n_2 - \#|D| < g + n - 1 = \dim \mathrm{RHS} \Rightarrow \exists(\omega, c)$.

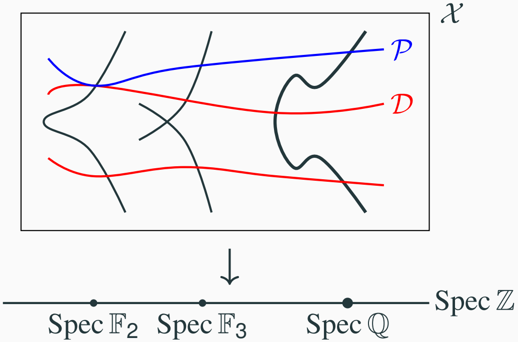
Goal: Compute ω and c.

D-intersection map

Note: $P \in Y(\mathbb{Q})$ lies in $\mathcal{Y}(\mathbb{Z}_S) \Leftrightarrow P$ does not reduce to a cusp modulo any $\ell \notin S$.

From now on: $S = \emptyset$ and $D(\mathbb{Q}) = D(\overline{\mathbb{Q}})$.

$$\begin{tikzcd}
b + \ker(\sigma) = \mathrm{Sel}(P_0, \Sigma) \arrow[r, hook] \arrow[d, "\sigma"] & J_Y(\mathbb{Q}) \arrow[d, "\sigma"] \\
\{b_0\} = \mathfrak{S}(P_0, \Sigma) \arrow[r, hook] & V_D
\end{tikzcd}$$



Main idea: combine the D-intersection map σ with p-adic integration.

Fundamental calculation

Goal: Compute the kernel of $\log_{J_Y}^* : \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D)) \to \ker(\sigma)^\vee$.

For $F = \sum_i x_i(F) G_i + \mathrm{div}(f) \in \mathrm{Div}^0(Y)$ and ω a log differential we have

$$\log_{J_Y}^*(\omega)(F) = \int_F \omega = \sum_{i=1}^{r} \left(\int_{G_i} \omega \right) x_i(F) + \int_{\mathrm{div}(f)} \omega$$
$$= \sum_{i=1}^{r} \left(\int_{G_i} \omega \right) x_i(F) + \sum_{Q \in D(\mathbb{Q})} \mathrm{Res}_Q(\omega) \log f(Q)$$

by the p-adic residue theorem.

If $F \in \ker(\sigma)$, get formula for prime factorisation of $f(Q)$.

Linear algebra

Look at $\log^*_{J_Y} : \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D)) \to \ker(\sigma)^\vee$.

- basis of LHS: $\omega_1, \ldots, \omega_g, \omega_{g+1}, \ldots, \omega_{g+n-1}$
- basis of RHS: $x_1, \ldots, x_r$ dual to MW basis G_i of $J(\mathbb{Q})$.

Matrix of $\log^*_{J_Y} : \mathrm{H}^0(X_{\mathbb{Q}_p}, \Omega^1(D)) \to \ker(\sigma)^\vee$ is $M(\Sigma) = \begin{pmatrix} A & B \\ \cancel{0} & \cancel{C} \\ \cancel{0} & \cancel{D(\Sigma)} \end{pmatrix}$

$$A_{i,j} = \int_{G_i} \omega_j$$

$$B_{i,j} = \int_{G_i} \omega_{g+j} - \sum_{Q \in D(\mathbb{Q})} \mathrm{Res}_Q\, \omega_{g+j} \sum_{\ell} i_{\mathcal{Q} \bmod \ell}(\Psi_\ell(G_i), \mathcal{Q}) \log(\ell)$$

Main result

Theorem (L.-Lüdtke, 2026)

Let $(\alpha_1, \dots, \alpha_{g+n-1}) \in \mathbb{Q}_p^{g+n-1}$ *be contained in the kernel of the matrix* $M(\Sigma)$ *and let* $\omega = \alpha_1\omega_1 + \cdots + \alpha_{g+n-1}\omega_{g+n-1}$*. Define*

$$c = \sum_{Q \in D(\mathbb{Q})} \mathrm{Res}_Q\, \omega \sum_{\ell} (-i_\ell(\mathcal{P}_0, \mathcal{Q}) + i_\ell(\Phi_\ell(\Sigma_\ell), \mathcal{Q})) \log(\ell).$$

Then

$$\int_F \omega = c \quad \text{for all } F \in \mathrm{Sel}(P_0, \Sigma).$$

Algorithm

Input: Curve $Y = X \setminus D$, base point P_0, set S, prime $p \notin S$

Output: Finite subset of $\mathcal{Y}(\mathbb{Z}_p)$ containing $\mathcal{Y}(\mathbb{Z}_S)$

1. Compute a Mordell–Weil basis G_i of $J(\mathbb{Q})$.
2. Find regular* model $\mathcal{X}$ of X and intersection matrix of $\mathcal{X}_\ell$.
3. Choose a basis ω_j of log differentials. Compute residues.
4. Calculate p-adic integrals of log differentials on X.
5. For each Σ, determine matrix $M(\Sigma)$, find a nontrivial element in the kernel, and compute the constant c.
6. On each residue disc, find the level set $\mathcal{Y}(\mathbb{Z}_p)_{\omega,c}$.

Goed verhaal, lekker kort

- $\mathcal{Y}(\mathbb{Z}_S)_\Sigma \subset$ a level set of the integral of a log differential ω.
- We can compute ω (and this level set) by computing
  - Coleman integrals of basic log differentials ω_j,
  - their residues,
  - and arithmetic intersection numbers with the cusps.

Thank you for your attention!