Computing Isomorphisms between Products of Supersingular Elliptic Curves

Joint work with P.-J. Spaenlehauer and P. Gaudry

Julien Soumier

ANTS, July 6, 2026

Contributions

Theorem: Deligne-Ogus-Shioda *[Shioda; 1979]*

Let $E_1, \ldots, E_g$ and $E'_1, \ldots, E'_g$ be supersingular elliptic curves ($g \geq 2$).
There exists an isomorphism:

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Algorithm to compute a Φ in polynomial time under GRH, given $\mathrm{End}(E_i), \mathrm{End}(E'_i)$.

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- Isomorphism of **unpolarized** abelian varieties.
- False for $g = 1$.

Motivation:

Post-quantum cryptography:

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Recent breakthrough in isogeny based cryptography.

$\Downarrow$

Toolbox needed to work with products of elliptic curves.

Contents

1. Elliptic curves and Deuring correspondence

2. The Deligne-Ogus-Shioda theorem

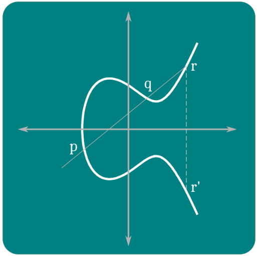
3. The strategy to compute an isomorphism

Elliptic curves and their morphisms

Elliptic curves:

Projective (abelian) algebraic group of dimension 1.

Figure: Example over $\mathbb{R}$.



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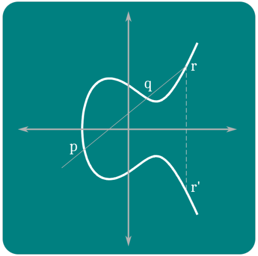

Morphisms of Elliptic Curves

A morphism $\varphi : E \to E'$ is:

- A morphism of varieties,
- A group homomorphism: $\varphi(P +_E Q) = \varphi(P) +_{E'} \varphi(Q)$.

Supersingular elliptic curves

Elliptic curve E over a field k

$\mathbb{Z} \subset \mathrm{End}(E)$

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Supersingular elliptic curve E over $\mathbb{F}_{p^n}$

$\mathrm{End}(E)$ is noncommutative,
of rank 4

Deuring correspondence

Theorem: Structure of Endomorphism Rings

Let E be a supersingular elliptic curve.

$$\mathrm{End}(E) \text{ is a maximal order in the quaternion algebra } \mathbb{Q}_{p,\infty}.$$

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Example when $p \equiv 3 \bmod 4$:

$E_0 : y^2 = x^3 + x, \quad (j(E_0) = 1728).$

$$\mathcal{O}_0 = \mathbb{Z} + i\,\mathbb{Z} + \tfrac{1+i\pi}{2}\,\mathbb{Z} + \tfrac{i+\pi}{2}\,\mathbb{Z},$$

with $i^2 = -1$.

Algorithmic consequences

Hard geometric problems	**Easy algebraic problems**
Compute morphisms $\varphi : E_1 \to E_2$.	$\mathbb{Z}$-linear algebra over quaternions

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(GRH) Proposition: *[Kohel/Lauter/Petit/Tignol; 2014], [Wesolowski; 2021]*

Given $\mathrm{End}(E_i) \simeq \mathcal{O}_i \subset \mathbb{Q}_{p,\infty}$, we can compute morphisms $E_1 \to E_2$.

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Given $\mathrm{End}(E_i) \simeq \mathcal{O}_i \subset \mathbb{Q}_{p,\infty}$, we can compute morphisms $E_1 \to E_2$.

$$\text{Compute } \alpha \in \mathrm{End}(E) \text{ such that } \deg(\alpha) = n$$
$$\iff$$
$$\text{Compute } a \in \mathcal{O} \text{ such that } \mathrm{Nrd}(a) = n$$

Our approach: toy example

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Cornacchia's algorithm is efficient in $\mathcal{O}_0$
thanks to low-discriminant subring.

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2. **The Deligne-Ogus-Shioda theorem**

3. The strategy to compute an isomorphism

Superspecial Abelian Varieties

Definition-Proposition: Product of Elliptic Curves

Let $E_1, \dots, E_g$ be elliptic curves.

- The product $E_1 \times \cdots \times E_g$ is still a variety and a group.
- A morphism $E_1 \times \cdots \times E_g \to E_1' \times \cdots \times E_g'$ is a morphism of variety and group.

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Purpose of our work: Compute Φ such that $\deg(\Phi) = 1$.

Reducing to dimension 2

Representation

g-dimensional morphism
$\Phi : E_1 \times \cdots \times E_g \to E'_1 \times \cdots \times E'_g$

$$\Longleftrightarrow \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1g} \\ \vdots & \ddots & \vdots \\ \varphi_{g1} & \cdots & \varphi_{gg} \end{pmatrix}$$

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If $g = 2k$:

- Compute $\Phi_i : E_{2i-1} \times E_{2i} \to E'_{2i-1} \times E'_{2i}$.
- Glue them $\mathrm{diag}(\Phi_1, \ldots, \Phi_k)$.

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If $g = 2k+1$, need the 3×3 case:

Compose:

$$E_1 \times E_2 \times E_3 \xrightarrow{(\Phi_1, \mathrm{Id}_{E_3})} E'_1 \times E'_2 \times E_3$$
$$E'_1 \times E'_2 \times E_3 \xrightarrow{(\mathrm{Id}_{E'_1}, \Phi_2)} E'_1 \times E'_2 \times E'_3$$

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Proposition: Slight improvement of *[Kani; 2014]* formula

Denote $d_{ij} := \deg(\varphi_{ij})$.

$$(\mathrm{Eq}_{\deg}) \quad \deg(\Phi) = (d_{11} + d_{21})(d_{12} + d_{22}) - \deg(\widehat{\varphi}_{12}\varphi_{11} + \widehat{\varphi}_{22}\varphi_{21}).$$

The underlying algorithmic problems

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$\Longrightarrow$

Given $\mathrm{End}(E_i)$, compute $(\varphi_{ij} : E_j \to E_0)$ satisfying $(\mathrm{Eq}_{\mathrm{deg}})$

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Completing matrices

Question: Given $\varphi_{11} : E_1 \to E_1'$ and $\varphi_{12} : E_2 \to E_1'$ compute $\varphi_{12}, \varphi_{22}$ such that $\begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} : E_1 \times E_2 \to E_1' \times E_2'$ is an isomorphism.

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- Algebraic conditions on $\varphi_{11}, \varphi_{12}$ so that $\varphi_{12}, \varphi_{22}$ exist.

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- Algebraic conditions on $\varphi_{11}, \varphi_{12}$ so that $\varphi_{12}, \varphi_{22}$ exist.
- An algorithm that given $\mathrm{End}(E_1), \mathrm{End}(E_1'), \mathrm{End}(E_2)$ and $\varphi_{11}, \varphi_{12}$ compute $\varphi_{12}, \varphi_{22}$ at the cost of fixing E_2'.

Computing $E_1 \times E_2 \to E_0 \times E_0$

Input: E_1, E_2 and their endomorphism rings.

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Compute an isomorphism $E_1 \times E_2 \to E_0 \times E$,
for some curve E over which we have no control.

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Ingredients:

- Use the Deuring correspondence to compute $\varphi_{11}, \varphi_{12}$.
- Apply previous algorithm to recover $\varphi_{21}, \varphi_{22}$ fixing E.

Computing : $E_1 \times E_2 \to E_0 \times E_0$

Input: E_1, E_2 and their endomorphism rings.

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Ingredients:

- Rewrite (Eq_{deg}) as $\mathrm{Nrd}(d_{21}\mu - d_{11}\nu) = d_{11}d_{21}$, $\mu, \nu \in \mathcal{O}_0$, Cornacchia's algorithm.
- ℓ-adic algorithms for $\mathcal{M}_2(\mathbb{Z}_\ell) \simeq \mathrm{End}(E_0) \otimes \mathbb{Z}_\ell$ at a prime $\ell \neq p$.

Conclusion

Isomorphisms
$E_1 \times E_2 \to E_0 \times E_0$

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Let $g \geq 2$. Given supersingular elliptic curves, $E_1, \ldots, E_g$ and $E_1', \ldots, E_g'$ along with their endomorphism rings, we can compute in polynomial time an isomorphism:

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Can we compute isomorphisms between more general abelian varieties ?

Generalizations

Let $A/\mathbb{F}_{p^2}$ such that $A \simeq E^g$, for E a supersingular elliptic curve and $g \geq 2$.

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- *[Page, Robert, S.; 2026]* solves the general case,
  $\Rightarrow$ given the module represention of A.

Generalizations

Let $A/\mathbb{F}_{p^2}$ such that $A \simeq E^g$, for E a supersingular elliptic curve and $g \geq 2$.

How can we compute $\varphi\colon A \simeq E^g$?

- *This work* solves the case $A = E_1 \times \cdots \times E_g$,
  $\Rightarrow$ with known $\mathrm{End}(E_i)$.
- *[Montessinos; 2026]* solves the case for any principally polarized abelian surface,
  $\Rightarrow$ with known IKO matrix.
- *[Page, Robert, S.; 2026]* solves the general case,
  $\Rightarrow$ given the module represention of A.

Thank you for your attention!