Computing class groups and gonalities of algebraic curves over finite fields

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Code available at: `https://github.com/nt-lib/CurveArith/`

Notation

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Definition

Let $k(X)$ be the function field of X. The gonality of X is

$$\operatorname{gon} X = \min_{f \in k(X) \setminus k} \deg(f).$$

Divisor enumeration

Given a divisor $D \in \operatorname{Div} X$, the Riemann-Roch space of D is defined by

$$\mathcal{L}(D) = \{0\} \cup \{f \in k(X)^\times : v_x(f) \geq -v_x(D) \quad \forall x \in X\}.$$

This is a k-vector space, and we denote by $\ell(D)$ its dimension.

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We have

$$\begin{aligned} \operatorname{gon}(X) &= \min_{f \in k(X) \setminus k} \deg(f) \\ &= \min_{\substack{D \geq 0 \\ \ell(D) \geq 2}} \deg(D). \end{aligned}$$

Divisor enumeration

for $d = 1, 2, \ldots$ **do**
    for all $D \geq 0, \deg(D) = d$ **do**
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The number of divisors which need to be considered grows exponentially with the gonality of X. However, this approach is still viable for curves of moderate genus over small fields, and is used for instance by Derickx-van Hoeij (2014), Derickx-Sutherland (2017), Najman-Orlić (2024) and Orlić (2026).

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Instead, we exploit the redundancy that occurs when computing large numbers of Riemann-Roch spaces.

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If $D = n_1 x_1 + \cdots + n_r x_r$ for some distinct $x_i \in X$, $n_i \in \mathbb{Z}_{\geq 1}$, then

$$\mathcal{L}(D_0 - D) = \mathcal{L}(D_0 - n_1 x_1) \cap \cdots \cap \mathcal{L}(D_0 - n_r x_r).$$

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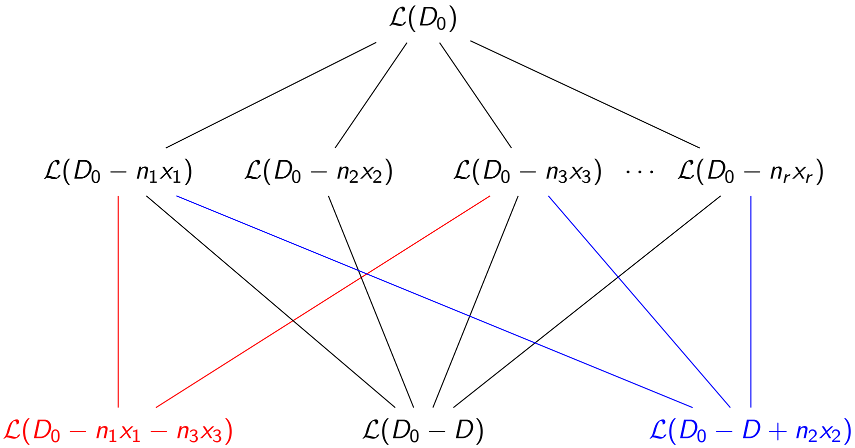
$$\mathcal{L}(D_0 - n_1 x_1) \qquad \mathcal{L}(D_0 - n_2 x_2) \qquad \mathcal{L}(D_0 - n_3 x_3) \quad \cdots \quad \mathcal{L}(D_0 - n_r x_r)$$

Computing Riemann-Roch spaces

$$\begin{tikzcd}[column sep=small]
 & & \mathcal{L}(D_0) \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[drr, dash] & & \\
\mathcal{L}(D_0 - n_1x_1) \arrow[drr, dash] & \mathcal{L}(D_0 - n_2x_2) \arrow[dr, dash] & \mathcal{L}(D_0 - n_3x_3) \arrow[d, dash] & \cdots & \mathcal{L}(D_0 - n_rx_r) \arrow[dll, dash] \\
 & & \mathcal{L}(D_0 - D) & &
\end{tikzcd}$$

Computing Riemann-Roch spaces



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- $\ker A \cap \ker B = \ker(A \mid B)$.
- Requires polynomial and ideal operations.

Computing gonalities

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$$D_0 = \sum_{\substack{x \in X \\ \deg(x) \leq d}} \left\lfloor \frac{d}{\deg(x)} \right\rfloor x.$$

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Instead, we use the Riemann-Roch theorem:

$$\ell(D) - \ell(K_X - D) = \deg(D) - g + 1,$$

where K_X is a canonical divisor on X.

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where K_X is a canonical divisor on X. Note that $\ell(K_X) = g(X)$.

Timings

Najman-Orlić (2024) compute the gonalities of 65 different modular curves $X_0(N)$ over various finite fields.

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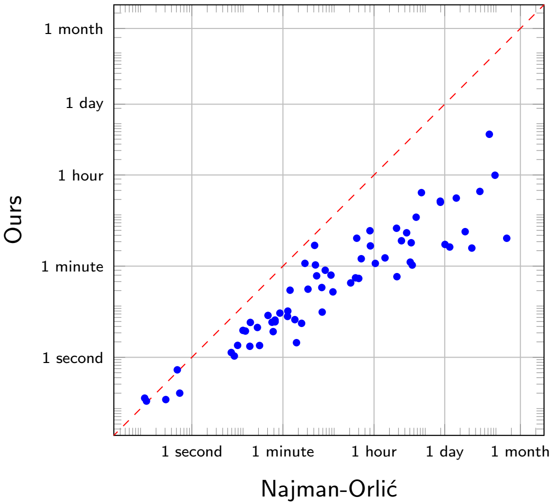

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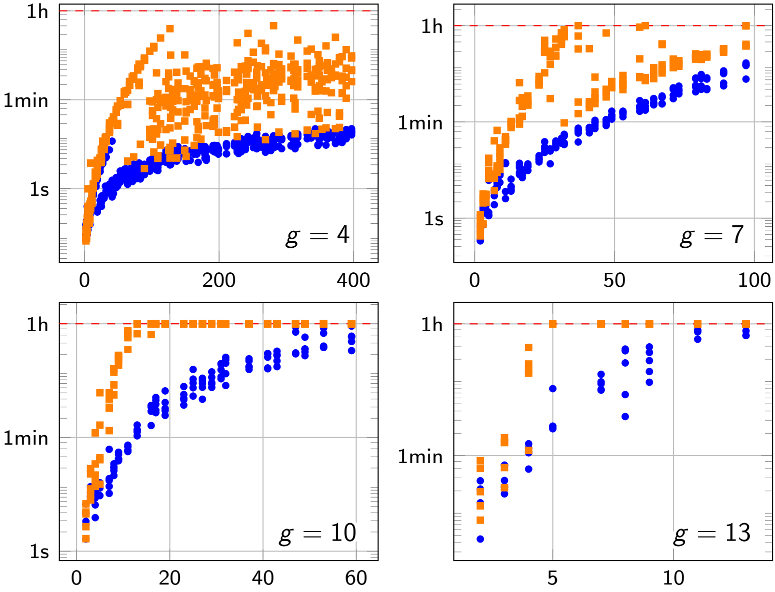
- Requires, in part, computing Riemann-Roch spaces $\mathcal{L}(D)$ for many (nearly) effective divisors D.
- As for gonalities, we need to transform the set of divisors for which we compute Riemann-Roch spaces to apply our linear algebraic procedure.

Timings


1h
1min
1s
0
200
400
$g = 4$
1h
1min
1s
0
50
100
$g = 7$
1h
1min
1s
0
20
40
60
$g = 10$
1h
1min
5
10
$g = 13$

Thank you for listening!