Chasing Rabbits Through Hypercubes: Better algorithms for higher dimensional 2-isogeny computations

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HD isogenies in cryptography

- Since 2022, an overwhelming majority of new isogeny based schemes use 2D isogenies.
- Some of them require 4D isogenies (SQIsignHD, qt-Pegasis, MIKE).
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✓ Possible using theta coordinates [Mum66].

Level 2 theta coordinates for isogeny computations

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- Let $e_n : A[n] \times A[n] \to \overline{k}^*$ be the n-th Weil pairing.
- A (ζ-)symplectic basis $\mathscr{B} := (S_1, \cdots, S_g, T_1, \cdots, T_g)$ of $A[n] \subseteq A(\overline{k})$ satisfies:

$$\forall l, m \in [\![1 \ ; \ g]\!], \quad e_n(S_l, S_m) = e_n(T_l, T_m) = 1,$$
$$\text{and} \quad e_n(S_l, T_m) = \zeta^{\delta_{l,m}},$$

where $\delta_{l,m} = 1$ if $l = m$ and 0 otherwise.

Definition: level 2 theta structures

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Let A/k be a PPAV of dimension g. A *theta structure* of level 2 is a map

$$\begin{array}{rcl} \theta^A : A & \longrightarrow & \mathbb{P}^{2^g-1} \\ P & \longmapsto & \theta^A(P) = (\theta^A_{\mathsf{i}}(P))_{\mathsf{i}\in(\mathbb{Z}/2\mathbb{Z})^g} \end{array}$$

along with a symplectic basis $\mathscr{B} := (S_1, \cdots, S_g, T_1, \cdots, T_g)$ of $A[2]$ satisfying the *theta group action relation*:

$$\theta^A_{\mathsf{i}}(P + S_m) = \theta^A_{\mathsf{i}+\mathsf{e}_m}(P) \quad \text{and} \quad \theta^A_{\mathsf{i}}(P + T_m) = (-1)^{i_m}\theta^A_{\mathsf{i}}(P),$$

for all $P \in A$, $\mathsf{i} \in (\mathbb{Z}/2\mathbb{Z})^g$ and $m \in [\![1\ ;\ g]\!]$, where $\mathsf{e}_m := (\delta_{l,m})_{1\le l\le g}$.

Properties of level 2 theta structures

Theorem (Mumford, 1966)

A level 2 theta structure $\Theta_A := (\theta^A, \mathscr{B})$ is fully determined by a symplectic basis $\mathscr{B}'$ of $A[4]$ such that $[2]\mathscr{B}' = \mathscr{B}$.

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Theorem

If A is not a (polarised) product, a level 2 theta structure induces an embedding of the Kummer variety:

$$\theta^A : A/\pm \hookrightarrow \mathbb{P}^{2^g-1}$$

- Points are represented up to sign: $\theta^A(P) = \theta^A(-P)$ for all $P \in A$.
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- This is the price to pay for minimising the number of coordinates.
- We have a differential addition and doubling formulae

$$(\theta^A(P), \theta^A(Q), \theta^A(P-Q)) \mapsto \theta^A(P+Q) \qquad \theta^A(P) \mapsto \theta^A([2]P).$$

Motivation: computing chains of 2-isogenies

Our working example: qt-Pegasis [DEIV25]

Inputs:

- Let $E/\mathbb{F}_p$ be a supersingular elliptic curve (p prime $\equiv 7 \mod 8$).
- Assume E is oriented by $\mathbb{Z}[(\sqrt{-p}+1)/2] \hookrightarrow \mathrm{End}(E)$ where $\sqrt{-p}$ is mapped to $\pi_p : (x, y) \in E \mapsto (x^p, y^p) \in E$.
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- Compute a corresponding 2^e-isogeny in dimension $g = 4$:

$$F : E^4 \to E_{\mathfrak{a}} \times E_{\overline{\mathfrak{a}}} \times S$$

A 2^e-isogeny satisfies $\widetilde{F} \circ F = [2^e]$, where $\widetilde{F} : E_{\mathfrak{a}} \times E_{\overline{\mathfrak{a}}} \times S \to E^4$ is the *contravariant* of F.

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- Extract $E_\mathfrak{a}$ from the codomain of F.

Computing a 2^e-isogeny: example of qt-Pegasis

Goal: Compute a 2^e-isogeny $F : A \longrightarrow B$ when given $T_1, \cdots, T_g \in A[2^{e+2}]$ such that $\ker(F) = \langle [4]T_1, \cdots, [4]T_g \rangle$. The computation is done in level 2 theta coordinates:

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Method (ex. qt-Pegasis, $g = 4$): Decompose F into a chain of 2-isogenies of length e:

$$A = E^4 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \quad \cdots \quad A_{e-2} \xrightarrow{f_{e-1}} A_{e-1} \xrightarrow{f_e} B = E_{\mathfrak{a}} \times E_{\overline{\mathfrak{a}}} \times S$$

Gluing — Generic isogenies — Splitting

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   $O(g \cdot e \log(e))$ point duplications and 2-isogeny evaluations needed.

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3. **Splitting:** Split the codomain B (into $E_{\mathfrak{a}} \times E_{\overline{\mathfrak{a}}} \times S$).

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- A level 2-theta structure Θ_A on A **adapted to** f *i.e.* induced by the symplectic basis $[2]\mathscr{B}$ of $A[4]$.

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Output:

- A level 2-theta structure Θ_B on B induced by the symplectic basis of $B[4]$:

$$f_*(\mathscr{B}) := ([2]f(S_1), \cdots, [2]f(S_g), f(T_1), \cdots, f(T_g)).$$

- This provides a way to evaluate f easily $\theta^A(P) \mapsto \theta^B(f(P))$.

Computing a generic 2-isogeny

Computation:

- Compute the **theta null-point** $\theta^B(0_B)$ from $\theta^A(T_1), \cdots, \theta^A(T_g)$.
- $\theta^B(0_B)$ determines (B, Θ_B) and $f : A \to B$.

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Evaluation: Use the **duplication formula**:

$$U^B(0_B) \odot U^B(f(P)) = H(\theta^A(P)^{\odot 2}),$$

with the notations:

- $x \odot y = (x_i \cdot y_i)_{i\in(\mathbb{Z}/2\mathbb{Z})^g}$ (dot product).
- $H(x) = \left(\sum_{j\in(\mathbb{Z}/2\mathbb{Z})^g}(-1)^{\langle i|j\rangle}x_j\right)_{i\in(\mathbb{Z}/2\mathbb{Z})^g}$ (Hadamard).
- $U^B(P) = H(\theta^B(P))$ (dual theta coordinates).

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Evaluation algorithm:

$$\theta^A(P) \xrightarrow{-^{\odot 2}} * \xrightarrow{H} * \xrightarrow{-\odot U^B(0_B)^{\odot -1}} * \xrightarrow{H} \theta^B(f(P)).$$

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- From $\theta^A(T_\mathsf{j})$, we know for all $\mathsf{i} \in (\mathbb{Z}/2\mathbb{Z})^g$,

$$\pi_{\mathsf{i},\mathsf{j}} := H(\theta^A(T_\mathsf{j})^{\odot 2})_\mathsf{i} \quad \text{and} \quad \pi_{\mathsf{i}+\mathsf{j},\mathsf{j}} := H(\theta^A(T_\mathsf{j})^{\odot 2})_{\mathsf{i}+\mathsf{j}}.$$

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- Propagation: from $U^B_\mathsf{i}(0_B)^{-1}$, we get

$$U^B_{\mathsf{i}+\mathsf{j}}(0_B)^{-1} = \frac{\pi_{\mathsf{i},\mathsf{j}}}{\pi_{\mathsf{i}+\mathsf{j},\mathsf{j}}} \cdot U^B_\mathsf{i}(0_B)^{-1} \quad \text{(by duplication formula)}$$

whenever $\pi_{\mathsf{i}+\mathsf{j},\mathsf{j}} \neq 0 \iff \pi_{\mathsf{i},\mathsf{j}} \neq 0$.

The hypercube inverse interpolation problem (HIIP)

Goal: Compute the projective inverse of $x := U^B(0_B)$.

Inputs:

- **Support** $S \subseteq (\mathbb{Z}/2\mathbb{Z})^g$ of indices of known $\theta^A(T_j)$.
  $S = \{e_1, \cdots, e_g\}$ when $\theta^A(T_1), \cdots, \theta^A(T_g)$ are known.
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Find $x^{\odot -1} \in \mathbb{P}^{2^g-1}(k)$ *such that:*

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⚠ Inversions are very costly so we solve the HIIP projectively with multiplications only.

The HIIP: a graph theoretic problem

The graph: $G = (V, E)$.

- Vertices $V := (\mathbb{Z}/2\mathbb{Z})^g$.
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- When $S = \{e_1, \cdots, e_g\}$, G is a subgraph of the hypercube.

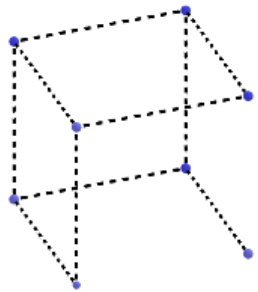

Figure: Subgraph of the (hyper)cube for $g = 3$ (with $\pi_{0,100} = \pi_{100,101} = 0$)

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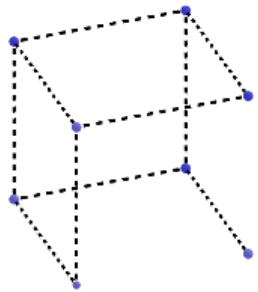

Figure: Subgraph of the (hyper)cube for $g = 3$ (with $\pi_{0,100} = \pi_{100,101} = 0$)

Theorem

The solution to the HIIP is unique iff G is connected.

Solving the HIIP

Solution to the HIIP: Use a covering subtree $\mathscr{T} \subseteq G$ to propagate values $x_i \to x_{i+j}$ through edges $(i+j, j) \in E$. We then have an explicit formula to compute $x^{\odot -1}$ with $\mathscr{T}$.

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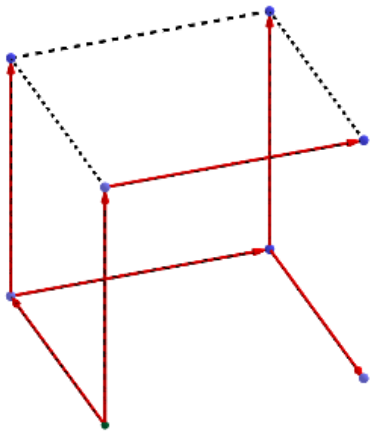
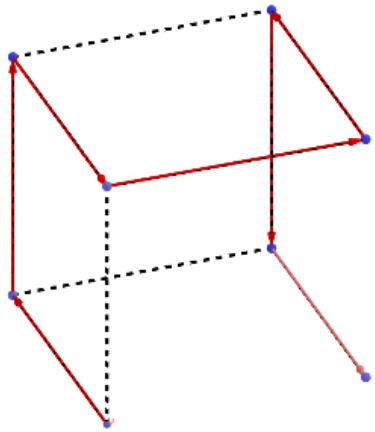
State of the art [Dar25], choose any covering tree $\mathcal{T}$.



✗ Found in variable time.

Our contribution: $\mathcal{T}$ is a Hamiltonian path.



✓ Found in constant time by iterating hypercube automorphisms.

Our contribution: solving the HIIP with a Hamiltonian path

- A Hamiltonian path is given by $\eta : [\![0\ ;\ 2^g-1]\!] \xrightarrow{\sim} (\mathbb{Z}/2\mathbb{Z})^g$ such that $(\eta(i), \eta(i+1)) \in E$ for all $i \in [\![0\ ;\ 2^g-2]\!]$.

$$\begin{array}{ccccccc} X_{\eta(0)} & & X_{\eta(1)} & & X_{\eta(2^g-2)} & & X_{\eta(2^g-1)} \\ \eta(0) & \longrightarrow & \eta(1) & \cdots & \eta(2^g-2) & \longrightarrow & \eta(2^g-1). \end{array}$$

Our contribution: solving the HIIP with a Hamiltonian path

- A Hamiltonian path is given by $\eta : [\![0\ ;\ 2^g-1]\!] \xrightarrow{\sim} (\mathbb{Z}/2\mathbb{Z})^g$ such that $(\eta(i), \eta(i+1)) \in E$ for all $i \in [\![0\ ;\ 2^g-2]\!]$.

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- Projectively, we have:

$$x_{\eta(i)}^{-1} = \left(\prod_{j=0}^{i-1} \pi_{\eta(j),s_j}\right) \cdot \left(\prod_{j=i}^{2^g-2} \pi_{\eta(j+1),s_j}\right) = \rho_L(i) \cdot \rho_R(i)$$

  with $s_j := \eta(j+1) - \eta(j)$.
- With iterative definitions $\rho_L(0) = 1$, $\rho_R(2^g-1) = 1$,

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- Algorithmic cost to solve the HIIP: $3(2^g-2)\mathrm{M}$ vs $(6 \cdot 2^g - 9)\mathrm{M}$ state-of-the-art.

Down the rabbit hole of gluing 2-isogenies

Onto gluing

$$\underbrace{E^4 = A \xrightarrow{f_1} A_1}_{\text{Gluing}} \underbrace{\xrightarrow{f_2} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f^{e-1}} A_{e-1}}_{\text{Generic}} \underbrace{\xrightarrow{f_e} B = E_{\mathfrak{a}} \times E_{\overline{\mathfrak{a}}} \times S}_{\text{Spliting}}$$

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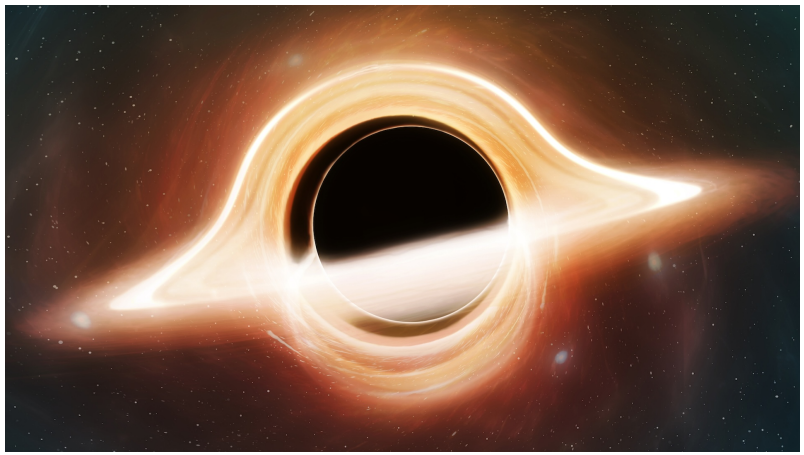

Figure: Artist view of a gluing isogeny

Image credit: Mark Garlick/Science Photo Library/Getty Images

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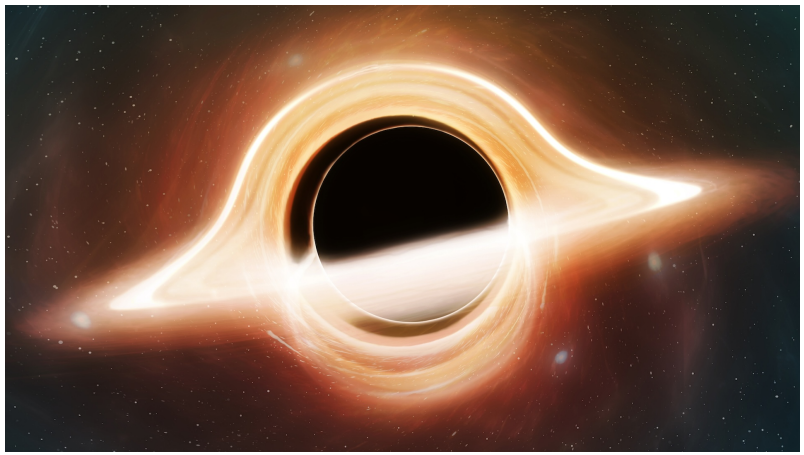

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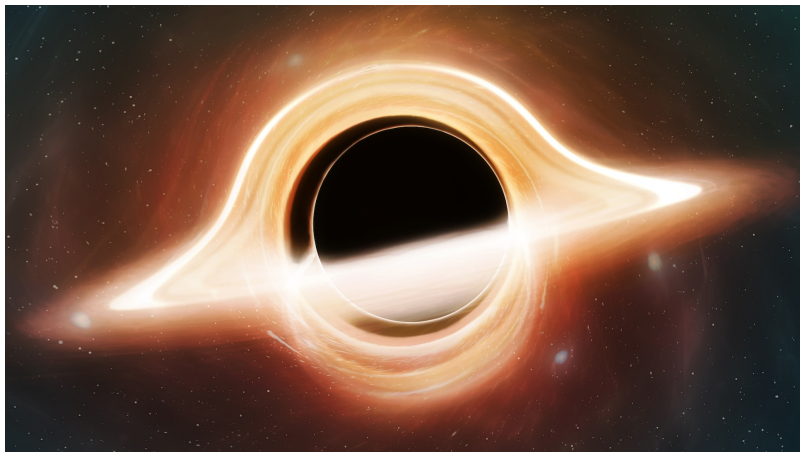
- Can be solved thanks to the graph theoretical PoV.



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On the generic covering tree computation

Goal: Finding valid Hamiltonian path

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1. Take Gray path $\mathscr{P}$

$v \in \mathbb{Z}_2^4$ and $i \circ \sigma := (i_{\sigma(m)})_{1 \leq m \leq 4}, \sigma \in \mathfrak{S}_4$

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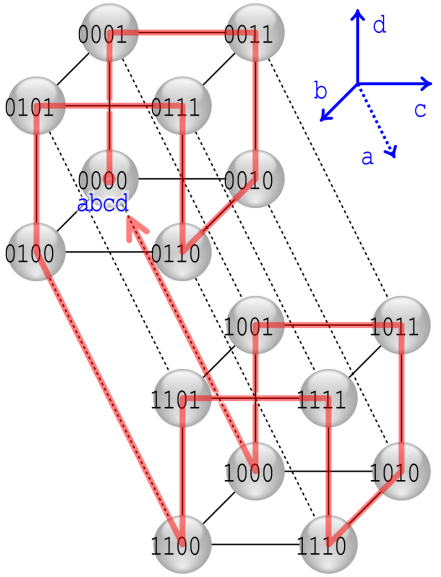

Figure: Gray Path $\mathscr{P}$ in H_4

On the generic covering tree computation

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Finding $\mathscr{T} \subset G$

1. Take Gray path $\mathscr{P}$
2. Use $\mathrm{Aut}(H_4)$:

$$\text{Find } \Delta_{\sigma,v} : i \in \mathbb{Z}_2^4 \longmapsto i \circ \sigma + v$$

such that $\Delta_{\sigma,v}(\mathscr{P}) \subset G$.

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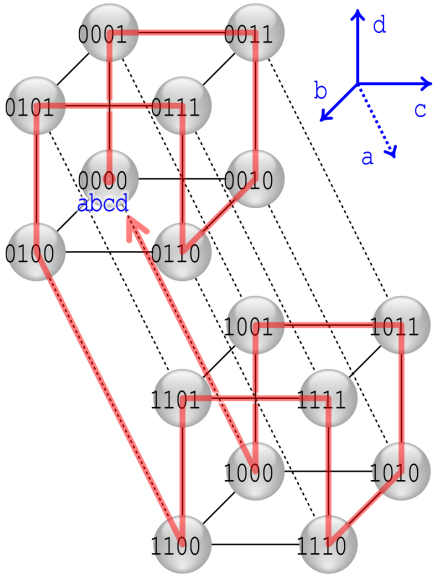

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▶ Working with $U^B(f(T))^{\odot -1}$, our gluing graphs are special.

Rabbits

In qt-Pegasis, gluing graphs are rabbits !

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Figure: A rabbit picture

Image credit: Izuzuki Diver: CC BY-CA 2.5

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Rabbit classification lemma

1. There are 32 rabbits, identical except for fur colour, of which there are 4.

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4. Under X-ray, rabbits are a long skeleton, with two ears on top.

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1. All rabbits are isomorphic.
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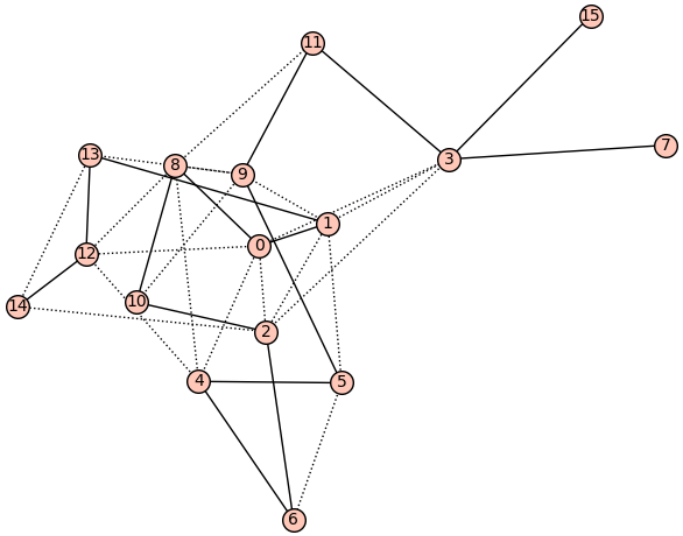

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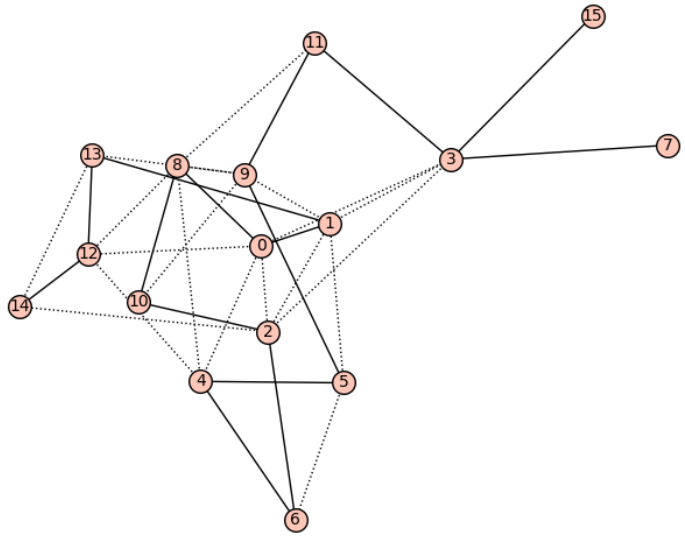

Figure: A rabbit picture

- By interpolating on rabbits, made constant time gluing.

Performance results

Comparison to state of the art

Thanks to our work, we gain:

- A constant time 2-isogeny chain.
- Faster kernel computation and gluing over $\mathbb{F}_p$.
- Faster generic codomain computations.

$\lceil \log_2(p) \rceil$	Old formulae (s)	New formulae (s)	Gain
505	0.757	0.613	19.0 %
1008	2.155	1.889	12.3 %
1525	4.516	4.013	12.3 %
2017	8.152	7.431	11.1 %
4030	41.426	39.073	5.7 %

Table: Comparison between new and old formulae in the SageMath implementation of qt-Pegasis. Average of 100 runs[2].

Timing captured on an AMD Ryzen 77840 U 8 cores CPU at 3.3 GHz, running Ubuntu 24.04.02 LTS, SageMath 10.6 and Python 3.12.11.

Constant time implementation of qt-Pegasis in C (4D chain)

$\lceil \log_2(p) \rceil$	p	ARM[1] (Mcycles)	X86[2] (Mcycles)	Timing[3] (ms)
505	$27 \cdot 2^{500} - 1$	146.95	274.38	23.94
1008	$15 \cdot 2^{1004} - 1$	871.34	1627.05	146.19
1525	$5 \cdot 2^{1522} - 1$	3163.05	5208.81	543.01
2017	$5 \cdot 2^{2014} - 1$	7041.82	11192.45	1222.96
4030	$45 \cdot 2^{4024} - 1$	55228.89	83757.68	10654.93

Table: Performance of the C-implementation of 4-dimensional chain of qt-Pegasis in CPU Mcycles. Results are the average of 100 benchmark runs.

1. Apple M1 Pro 8 cores CPU with nominal clock speed 3.2 GHz, running macOS 15.6.1 and using Clang v16.0.0.
2. Intel Xeon(R) CPU E5-1650 v3 6 cores with nominal clock speed 3.5 GHz, running Manjaro 5.10.237, and using gcc 15.1.1.
3. Apple M4 Pro CPU with nominal clock speed of 4.5 GHz, running macOS 26.2 and using Clang v17.0.0.

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Thanks for listening !

Beware of rabbits !

Image credit: Killer rabbit of Caerbannog, taken from the historical documentary "Monty Python and the Holy Grail".