Ordinary abelian varieties: Isogeny graphs and polarizations

joint work with Edgar Costa, Taylor Dupuy, David Roe, Christelle Vincent

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ANTS XVII

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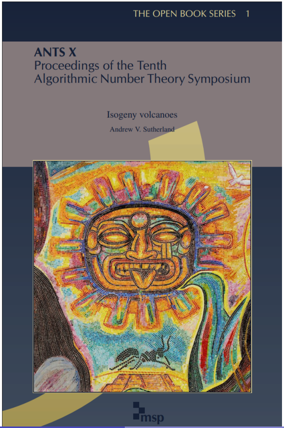

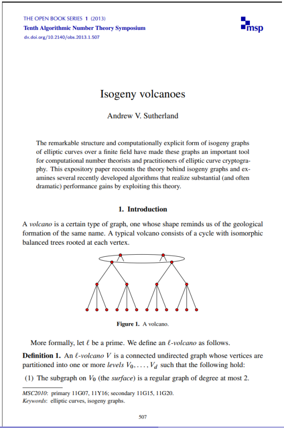
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Isogeny volcanoes

Andrew V. Sutherland

The remarkable structure and computationally explicit form of isogeny graphs of elliptic curves over a finite field have made these graphs an important tool for computational number theorists and practitioners of elliptic curve cryptography. This expository paper recounts the theory behind isogeny graphs and examines several recently developed algorithms that realize substantial (and often dramatic) performance gains by exploiting this theory.

1. Introduction

A *volcano* is a certain type of graph, one whose shape reminds us of the geological formation of the same name. A typical volcano consists of a cycle with isomorphic balanced trees rooted at each vertex.

Figure 1. A volcano.

More formally, let ℓ be a prime. We define an ℓ-*volcano* as follows.

Definition 1. An ℓ-*volcano* V is a connected undirected graph whose vertices are partitioned into one or more *levels* $V_0, \dots, V_d$ such that the following hold:

(1) The subgraph on V_0 (the *surface*) is a regular graph of degree at most 2.

MSC2010: primary 11G07, 11Y16; secondary 11G15, 11G20.
Keywords: elliptic curves, isogeny graphs.

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Consider the geometrically simple isogeny class with LMFDB label 4.3.c_ab_af_ai.

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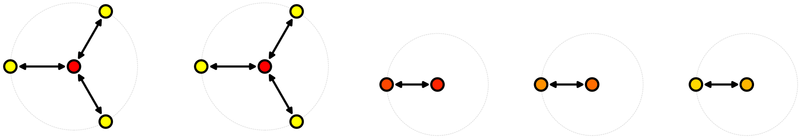
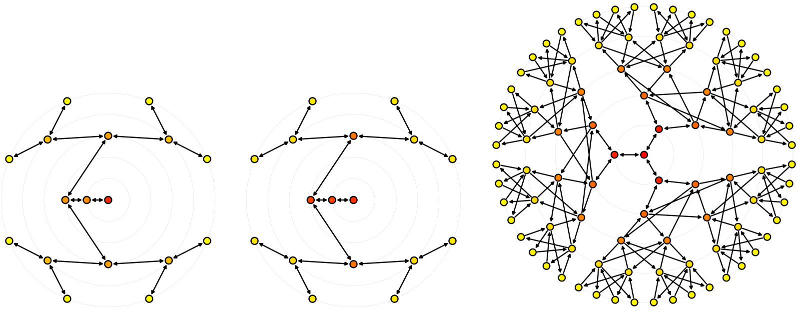
The 2-isogeny graph has 14 vertices, in 5 connected components, here colored by endomorphism ring



(within each component a darker color indicates a larger End).

Base change to $\mathbb{F}_9$: 1763 isomorphism classes. Its 2-isogeny graph has 35 components: two with 49 vertices, nine with 17, twelve with 94, and twelve with 32 vertices. Some of them (same convention on colors; no relation with the previous pictures):



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For any isogeny $\varphi\colon A \to B$ in $\mathscr{I}_h$, if $I = \mathscr{F}(A)$ and $J = \mathscr{F}(B)$ then there is an element $x \in (J : I) \cap K^\times$ such that $\mathscr{F}(\varphi) = x$.

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Example : minimal isogenies can make End's jump.

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The lattice of inclusions – all of index 2 – of the End's in $\mathscr{I}_h$ is:

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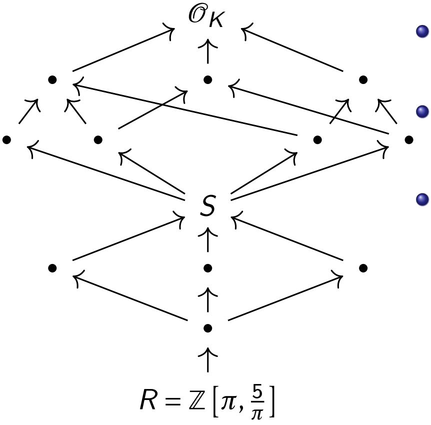

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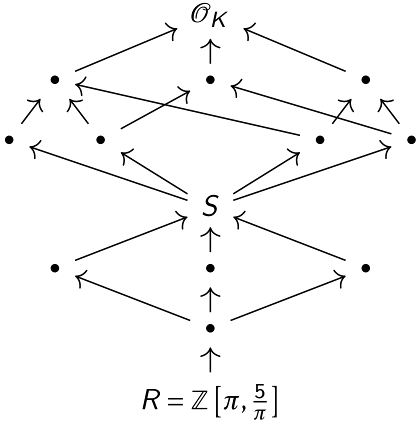

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- ... but the inclusion $S \subsetneq \mathscr{O}_K$ is not minimal!

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1. *φ is horizontal or vertical: $\mathrm{End}(A) \subseteq \mathrm{End}(B)$ or $\mathrm{End}(B) \subseteq \mathrm{End}(A)$.*
2. *If $\mathfrak{m}$ is regular then φ is horizontal.*
3. *If φ is vertical then $\mathfrak{m}$ is singular.*
4. *Let $T = \mathrm{End}(A) \cup \mathrm{End}(B)$ and $S = \mathrm{End}(A) \cap \mathrm{End}(B)$. Then $|T/S| = |R/\mathfrak{m}|^{\mathrm{length}_R(T/S)}$. In particular, φ is vertical if and only if $[R : \mathfrak{m}]$ divides the index $[T : S]$.*

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Definition

1. Two isogenies $\varphi_1 \colon A_1 \to B_1$ and $\varphi_2 \colon A_2 \to B_2$ are said to be equivalent, denoted $\varphi_1 \approx \varphi_2$, if and only if there exist isomorphisms $i_1 \colon A_1 \to A_2$ and $i_2 \colon B_1 \to B_2$ such that $i_2 \circ \varphi_1 = \varphi_2 \circ i_1$.

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2. The full isogeny graph $\mathscr{G}$ of an isogeny class $\mathscr{I}_h$ is the directed multigraph whose vertices are isomorphism classes of abelian varieties in $\mathscr{I}_h$ and whose edges are equivalence classes of isogenies of deg. > 1.

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1. Two isogenies $\varphi_1 \colon A_1 \to B_1$ and $\varphi_2 \colon A_2 \to B_2$ are said to be equivalent, denoted $\varphi_1 \approx \varphi_2$, if and only if there exist isomorphisms $i_1 \colon A_1 \to A_2$ and $i_2 \colon B_1 \to B_2$ such that $i_2 \circ \varphi_1 = \varphi_2 \circ i_1$.
2. The full isogeny graph $\mathscr{G}$ of an isogeny class $\mathscr{I}_h$ is the directed multigraph whose vertices are isomorphism classes of abelian varieties in $\mathscr{I}_h$ and whose edges are equivalence classes of isogenies of deg. > 1.
3. Let D be a positive integer. The D-isogeny graph $\mathscr{G}^D$ is the subgraph of $\mathscr{G}$ with edges $\mathscr{E}^D$ of degree dividing D.

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4. Let D and r be positive integers. The graph $\mathscr{G}^{D,r}$ is the subgraph of $\mathscr{G}$ with edges $\mathscr{E}^{D,r}$ of degree dividing D and length at most r.

Proposition

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$$\varphi_1 \approx \varphi_2 \iff \begin{array}{c} \exists k_A, k_B \in K^\times \text{ such that} \\ k_A I_1 = I_2,\ k_B J_1 = J_2 \text{ and} \\ x_2 k_A = k_B x_1 \end{array}$$

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Given, in $\mathscr{I}_h$

- $\varphi : A \to B \overset{\mathscr{F}}{\rightsquigarrow} x \cdot H \subseteq I$,
- $\psi : B \to C \overset{\mathscr{F}}{\rightsquigarrow} y \cdot I \subseteq J$,
- ρ *an automorphism of* $B \overset{\mathscr{F}}{\rightsquigarrow} z \in (I : I)^\times$,

then

$$\psi \circ \varphi \approx \psi \circ \rho \circ \varphi \iff z \in (H : H)^\times (J : J)^\times .$$

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- ... $\mathrm{Pic}(R)$ acts on edges too: if L is an invertible R-ideal and $x \cdot I \subseteq J$ is a (minimal) isogeny, then $x \cdot IL \subseteq JL$ is a (minimal) isogeny.
- So: need to enumerate min. isogenies only once per weak eq. class.

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- We have also the Magma intrinsics `IsogenyOrbitBuilder` in which the output is modulo the action of Pic on both source and target (independently). Useful for storing!

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   Then the isogeny graph $\mathscr{G}^D$ *is connected (as a directed graph).*
2. *Let* $\mathscr{S}$ *be as in (1) above and set* $S_{\mathscr{S}} = \sum_{\mathfrak{l} \in \mathscr{S}} \left(\mathrm{ord}_{\mathrm{Pic}(\mathcal{O}_K)}([\mathfrak{l}\mathcal{O}_K]) - 1\right)$*.*
   Then the diameter of $\mathscr{G}^{D,1}$ *satisfies:*

$$\mathrm{diam}(\mathscr{G}^{D,1}) \leq \begin{cases} 2\,\mathrm{length}_R(\mathcal{O}_K/R) + S_{\mathscr{S}} & \textit{if } R \textit{ is Bass,} \\ 2\left(\mathrm{length}_R(\mathcal{O}_K/\mathfrak{f}_R) - 1\right) + S_{\mathscr{S}} & \textit{otherwise.} \end{cases}$$

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$$N_{\mathscr{S}^{\mathrm{sing}}} = \mathrm{lcm}\left(\{|R/\mathfrak{l}| : \mathfrak{l} \in \mathscr{S}^{\mathrm{sing}}\}\right).$$

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4. *Assume that $\mathscr{G}^D$ is connected. Then*

$$\mathrm{length}_R(R/\mathfrak{f}_R) \leq \mathrm{diam}(\mathscr{G}^{D,1}).$$

Thank you for your attention!