Explicit Brauer-Manin Obstructions on Plane Quartics

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(joint work with Nils Bruin)

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Problem

Given a plane quartic curve C over $\mathbb{Q}$, how can we decide if $C(\mathbb{Q}) \neq \emptyset$?

Cases:

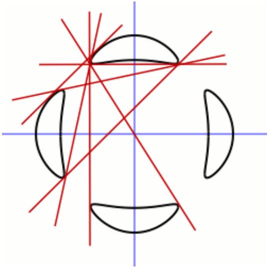
- When $C(\mathbb{Q}) \neq \emptyset$: usually easy to see this!
- When $C(\mathbb{Q}) = \emptyset$: can be more challenging to verify.

Main results:

- A method to prove $C(\mathbb{Q}) = \emptyset$, implemented in magma.
- Works well for small discriminant: resolves all but 13 of the 82,241 curves in Sutherland's database ANTSXIII[1].
- Can also be used to rule out points of small degree.
- Particular case of descent / Brauer-Manin obstruction (but proof of correctness is more elementary).

[1] The method was only really needed for 148 curves

Descent algorithm of Bruin-Poonen-Stoll



$L/\mathbb{Q}$	degree 28 bitangent algebra
$f \in L[x, y, z]$	generic bitangent
S	bad primes

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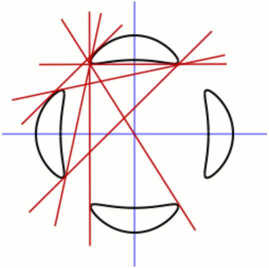

$$\begin{tikzcd}
C(\mathbb{Q}) \arrow[r] \arrow[d, "f"'] & \prod_{p\in S} C(\mathbb{Q}_p) \arrow[d, "\prod f_p"] \\
\left(\dfrac{L^\times}{\mathbb{Q}^\times L^{\times 2}}\right)_S \arrow[r] & \prod_{p\in S} \dfrac{L_p^\times}{\mathbb{Q}_p^\times L_p^{\times 2}}
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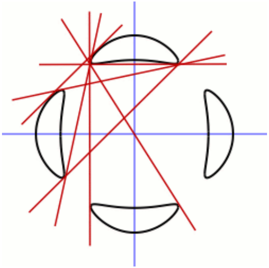

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Algorithm:

1. Compute $\mathcal{O}_{L,S}^\times$, $\mathrm{Cl}(\mathcal{O}_{L,S})[2]$ to get image of bottom map
2. Compute images of f_p maps for $p \in S$.
3. If intersection of these is empty, then $C(\mathbb{Q}) = \emptyset$.

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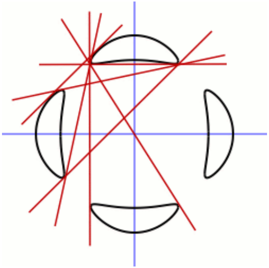

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$$\begin{array}{ccccc} C(\mathbb{Q}) & \longrightarrow & \prod_{p \in S} C(\mathbb{Q}_p) & & \\ \downarrow f & & \downarrow \prod f_p & & \\ \left(\frac{L^\times}{\mathbb{Q}^\times L^{\times 2}}\right)_S & \longrightarrow & \prod_{p \in S} \frac{L_p^\times}{\mathbb{Q}_p^\times L_p^{\times 2}} & \longrightarrow & \left(\frac{L^\times}{L^{\times 2}}\right)^*_{S,\mathrm{Norm}=1} \end{array}$$

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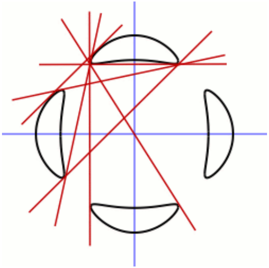

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Improved Algorithm:

1. Compute **some** square norm elements in L and their orthogonal complement in $\prod L_p^\times/\mathbb{Q}_p^\times L_p^{\times 2}$.
2. Compute images of f maps for $p \in S$.
3. If intersection of these is empty, then $C(\mathbb{Q}) = \emptyset$.

Example

$$C : x^4 + y^4 + x^2yz + 2xyz^2 - y^2z^2 + z^4 = 0$$

- Bruin-Poonen-Stoll: $C(\mathbb{Q}) = \emptyset$ conditionally on GRH.
- Bitangent algebra L has Minkowski bound 10^{22}.
- We find some square norm elements $B \subset L^\times$ and use this to show $C(\mathbb{Q}) = \emptyset$ unconditionally.

$$\begin{tikzcd}
C(\mathbb{Q}) \arrow[r] \arrow[d] & \prod_{p\in S} C(\mathbb{Q}_p) \arrow[d] & \\
\left(\frac{L^\times}{\mathbb{Q}^\times L^{\times 2}}\right)_S \arrow[r] & \prod_{p\in S} \frac{L_p^\times}{\mathbb{Q}_p^\times L_p^{\times 2}} \arrow[r] & B^*
\end{tikzcd}$$

Computing elements of square norm unramified outside S

- Generate random $\alpha_i \in L^\times$ supported on some factor basis to get a row for a 'relation matrix'

$$[\ \underbrace{\mathrm{ord}_{\mathfrak{p}_1}(\alpha_i)\quad \cdots \quad \mathrm{ord}_{\mathfrak{p}_N}(\alpha_i)}_{\text{factor basis omitting } \mathfrak{p}_i \mid S}\ \underbrace{\mathrm{ord}_{p_1}(N(\alpha_i))\quad \cdots \quad \mathrm{ord}_{p_n}(N(\alpha_i))}_{\text{primes of } \mathbb{Q} \text{ below factor basis}}\ ]$$

- Left kernel of this matrix modulo 2 gives products of the α_i which have square norm and even valuation outside S.

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We don't need relations to generate the full unit group!

Computing the index of plane quartic curves

$\mathrm{Index}(C) = GCD$ of degrees of closed points

- $\mathrm{Index} \mid 4$
- $\mathrm{Index} = 1 \Leftrightarrow$ rational or cubic points $\Leftrightarrow \mathrm{Pic}^1(C) \neq \emptyset$
- $\mathrm{Index} = 2 \Rightarrow$ quadratic or sextic points $\Leftrightarrow \mathrm{Pic}^2(C) \neq \emptyset$
- $\mathrm{Index}(C_{\mathbb{Q}}) \geq \max_p(\mathrm{Index}(C_{\mathbb{Q}_p}))$

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We can use a similar approach to get better bound

$$\begin{tikzcd}
\text{Pic}^i(C) \arrow[r] \arrow[d] & \prod_{p \in S} \text{Pic}^i(C_{\mathbb{Q}_p}) \arrow[d] & \\
\left(\frac{L^\times}{\mathbb{Q}^\times L^{\times 2}}\right)_S \arrow[r] & \prod_{p \in S} \frac{L_p^\times}{\mathbb{Q}_p^\times L_p^{\times 2}} \arrow[r] & B^*
\end{tikzcd}$$

Examples

Example (a quartic of index 4)

$$x^4 + x^3y + 4x^2y^2 + 2xy^3 + 3y^4 + 2x^3z + 2x^2yz + 2xy^2z$$
$$+ 3y^3z + x^2z^2 + 7xyz^2 + 3y^2z^2 - 4xz^3 - 6yz^3 + 2z^4$$

has index ≤ 2 everywhere locally, but index 4 over $\mathbb{Q}$.

Example (a quartic of index 2)

$$x^4 + 3x^2y^2 - xy^3 + 3y^4 + 3x^3z + 3x^2yz + 4xy^2z + 6y^3z$$
$$- x^2z^2 + 5xyz^2 - 2y^2z^2 - 4xz^3 - 5yz^3 + 2z^4 .$$

Has index ≤ 1 everywhere locally, but index 2 over $\mathbb{Q}$.

Relation to descent and BM

Our obstruction involves

- A set of primes S
- A subset $B \subset (\mathcal{O}_{L,S}^{\times}/\mathcal{O}_{L,S}^{\times 2})_{N=1}$

Theorem

There is an explicitly computable finite set of primes S such that with $B = (\mathcal{O}_{L,S}^{\times}/\mathcal{O}_{L,S}^{\times 2})_{N=1}$, the following are equivalent

1. $V_{B,S}(C) = \emptyset$, *our obstruction*
2. $C(\mathbb{A}_k)^{\mathrm{Br}_{\lambda_\Delta}(C)} = \emptyset$, *BMO from a particular subgroup*
3. $C(\mathbb{A}_k)^{\lambda_\Delta} = \emptyset$, *Descent obstruction of particular type*
4. $\mathrm{Sel}^f(C) = \emptyset$. *fake descent obstruction of BPS*

Summary:

- We give a practical algorithm to compute obstructions to rational (or low degree points) on plane quartic curves.
- In principle this works for any curve (not just quartics)
- Improves upon earlier descent based approach which required full class/unit group information.

Possible improvements to implementation:

- Make better use of the fact that full relation matrices are not needed.
- Local images (particularly for higher degree points).