$K3$ surfaces and arithmetic - results and questions

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K3 surfaces

X complex $K3$ surface

2-dimensional complex manifold

compact, connected, simply connected, $K_X = 0$.

name $K3$: André Weil, 1958

$K + K + K$: Kummer, Kähler, Kodaira

André Weil and K3

"Dans la seconde partie de mon rapport, il s'agit des variétés kählériennes de type $K3$, ainsi nommées en l'honneur de

Kummer, Kähler, Kodaira,

et de la belle montagne $K2$."

Hodge decomposition

$$\dim(H^2(X, \mathbf{C})) = 22.$$

$$H^2(X, \mathbf{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}.$$

$$\dim(H^{2,0}) = \dim(H^{0,2}) = 1.$$

$$\dim(H^{1,1}) = 20.$$

Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The $K3$ lattice

$$L_X = H^2(X, \mathbf{Z})$$

L_X is a free $\mathbf{Z}$-module or rank 22.

$$q : L_X \times L_X \to \mathbf{Z} : \text{ cup product.}$$

(L_X, q) is an even unimodular lattice, of signature $(3, 19)$.

$$(L_X, q) \simeq (-E_8) \oplus (-E_8) \oplus U \oplus U \oplus U.$$

$$L_X : K3\text{-lattice}$$

DYNAMICAL DEGREES OF AUTOMORPHISMS

X complex $K3$ surface

$a : X \to X$ an automorphism

induces an isometry

$$a^* : L_X \to L_X .$$

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$$a^* : L_X \to L_X.$$

Dynamical degree of a is by definition the spectral radius of a^*.

X complex $K3$ surface

$a : X \to X$ an automorphism

Dynamical degree of a :

1 or a Salem number.

McMULLEN

$a : X \to X$ an automorphism

Characteristic polynomial of a^* :

$$S.C$$

$S = 1$ or a Salem polynomial.

$C = 1$ or a product of cyclotomic polynomials.

SALEM POLYNOMIAL

$S \in \mathbf{Z}[X]$, monic, irreducible, symmetric, $\deg(S) \geqslant 4$

is a Salem polynomial if

S has exactly two roots not on S^1, both positive real numbers.

α, α^{-1}

$\alpha > 1$ is called Salem number.

Symmetric (reciprocal) : $S(X) = X^{\deg(S)} S(X^{-1})$.

Example

Lehmer polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

Corresponding Salem number : Lehmer number

$$\lambda_{10} = 1{,}17628...$$

smallest known Salem number.

McMULLEN'S QUESTION

rank of $L_X = 22$

$a : X \to X$ an automorphism

Dynamical degree of a is 1 or a

Salem number of degree $\leqslant 22$.

Question. (McMullen, 2002) :

Which Salem numbers occur ?

REALIZABLE

Let α be a Salem number of degree $d \leqslant 22$. We say that α is realizable if there exist

a complex $K3$ surface X

and an automorphism $a : X \to X$

of dynamical degree α.

Results - algebraic dimension

Results depend on the algebraic dimension of the surface.

X complex $K3$ surface

$alg(X)$: transcendence degree of the field of meromorphic functions on X.

$$alg(X) = 0,\ 1 \text{ or } 2.$$

X has an automorphism of with dynamical degree $> 1 \implies$

$$alg(X) = 0 \text{ or } 2.$$

$$alg(X) = 2 \iff X \text{ is projective.}$$

McMullen, Oguiso, E.B.

Lehmer number λ_{10} is realizable for surfaces X with $alg(X) = 0$ and also with $alg(X) = 2$.

The second smallest known Salem number λ_{18} is realizable for surfaces X with $alg(X) = 2$ but not with $alg(X) = 0$.

The third smallest known Salem number λ_{14} is realizable for surfaces X with $alg(X) = 0$ but not with $alg(X) = 2$.

$$\lambda_{10} = 1,17628...,\ \lambda_{18} = 1,18836...,\ \lambda_{14} = 1,20002....$$

Results for $alg(X) = 0$

Let α be a Salem number of degree 22

associated Salem polynomial S

Gross-McMullen (2002)

α realizable $\implies$ $|S(1)|$ and $|S(-1)|$ are squares.

Results for $alg(X) = 0$

Let α be a Salem number of degree 22

associated Salem polynomial S

Gross-McMullen (2002)

$$\alpha \text{ realizable} \implies |S(1)| \text{ and } |S(-1)| \text{ are squares.}$$

B. - Taelman (2020) :

$$\alpha \text{ realizable} \iff |S(1)| \text{ and } |S(-1)| \text{ are squares.}$$

Results for $alg(X) = 0$

Let α be a Salem number of degree $d \leqslant 20$.

- $d \equiv 0 \pmod 4$

  α is realizable.

- $d \equiv -2 \pmod 8$

  α is realizable.

PROOFS

B. (2024, 2026), Takada (2023) : Isometries of unimodular lattices with a given characteristic polynomial.

Let

$$F \in \mathbf{Z}[X]$$

monic, and let

L be a unimodular lattice.

Does there exist

$$t \in O(L)$$

with characteristic polynomial F ?

PROOFS

$$F = S.C,$$

with

S : Salem polynomial

C : product of cyclotomic polynomials,

$$L : K3\text{-lattice}.$$

PROOFS

Hasse principle

- Local conditions.

- Obstruction to Hasse principle.

DEGREES 10 and 18

Let α be a Salem number of degree $d \leqslant 20$.

- $d \equiv 2 \pmod{8}$

Question :

$$d = 10 \implies \alpha \text{ is realizable ?}$$

Obstruction

Let

$$f, g \in \mathbf{Z}[X] \text{ be monic, symmetric.}$$

Let $V_{f,g}$ be the set of prime numbers p such that

$$f \pmod{p} \text{ and } g \pmod{p}$$

have a common symmetric irreducible factor in $\mathbf{F}_p[X]$.

$$F \text{ symmetric : } F(X) = X^{\deg(F)} F(X^{-1}).$$

EXAMPLE

$$S(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

$$\Phi(X) = X^2 + 1$$

$$3 \in V_{S,\Phi}.$$

DEGREE 10 - QUESTION

Let S be a Salem polynomial of degree 10 such that

$$|S(1)| = |S(-1)| = 1.$$

α : Salem number corresponding to S.

Question. Does there exist a cyclotomic polynomial Φ of degree $\leqslant 12$, $\Phi \neq \Phi_{13}, \Phi_{26}$, such that

$$V_{S,\Phi} \neq \varnothing \; ?$$

YES $\iff$ α is realizable.

DEGREE 18

Let S be a Salem polynomial of degree 18 such that

$$|S(1)| = |S(-1)| = 1.$$

α : Salem number corresponding to S.

Question. Does there exist a cyclotomic polynomial Φ of degree $\leqslant 4$, $\Phi \neq \Phi_5, \Phi_{10}$, such that

$$V_{S,\Phi} \neq \varnothing \ ?$$

YES $\iff$ α is realizable.

DEGREE 18

$$\lambda_{18} = 1,2562...$$

is NOT realizable.

$$S(X) = X^{18} - X^{17} + X^{16} - X^{15} - X^{12} + X^{11} - X^{10} + \\ + X^9 - X^8 + X^7 - X^6 - X^3 + X^2 - X + 1.$$

$$V_{S,\Phi} = \varnothing$$

For all cyclotomic polynomials Φ of degree $\leqslant 4$, $\Phi \neq \Phi_5, \Phi_{10}$.

OPEN QUESTIONS

Degree 10 : Is every Salem number of degree 10 realizable ?

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Degree 10 : Is every Salem number of degree 10 realizable ?

Degree 18 : Infinitely many examples ? Density ?

$\mathrm{alg}(X) = 2$: ??

McMullen (2016). Mainly examples, computer assisted.

The invariants n_X and m_X

PICARD LATTICE

X complex $K3$ surface.

$$L_X = H^2(X, \mathbf{Z}), \text{ rank } 22, \text{ signature } (3, 19).$$

Hodge decomposition

$$H^2(X, \mathbf{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

Picard lattice

$$S_X = L_X \cap H^{1,1}(X).$$

$$\rho_X = \mathrm{rank}(S_X).$$

PICARD LATTICE

$$alg(X) = 2 \iff \mathrm{sign}(S_X) = (1, \rho_X - 1)$$

$alg(X) = 1 \iff S_X$ has a one-dimensional kernel

$$alg(X) = 0 \iff \mathrm{sign}(S_X) = (0, \rho_X).$$

$$Aut(X) \to O(L_X)$$

$$a \mapsto a^*$$

is injective. What about

$$Aut(X) \to O(S_X)$$

$$a \mapsto a^*|S_X \ ?$$

TRANSCENDENTAL LATTICE

Transcendental lattice

$$T_X = {S_X}^{\perp}$$

the orthogonal of S_X in the lattice L_X.

$$alg(X) = 2 \iff \mathrm{sign}(T_X) = (2, 20 - \rho_X)$$

$$alg(X) = 1 \iff T_X \text{ has a one-dimensional kernel}$$

$$alg(X) = 0 \iff \mathrm{sign}(T_X) = (3, 19 - \rho_X).$$

$$a \in Aut(X).$$

Action of a^* on T_X ?

$alg(X) = 2 \implies$ the characteristic polynomial of $a^*|T_X$ is a power of a cyclotomic polynomial.

$alg(X) = 1 \implies a^*|T_X$ is the identity,

$alg(X) = 0 \implies$ either $a^*|T_X$ is the identity, or the characteristic polynomial of $a^*|T_X$ is a Salem polynomial.

$alg(X) = 2$

From now on, assume that

$$alg(X) = 2.$$

$$f : Aut(X) \to O(T_X)$$

$$a \mapsto a^*|T_X$$

Set

$$Aut^0(X) = \mathrm{Ker}(f).$$

Exact sequence

$$1 \to Aut^0(X) \to Aut(X) \to M_X \to 1$$

with M_X finite, cyclic. Set

$$m_X = |M_X|.$$

$\varphi(m_X) \leqslant 20.$

$$Aut(X) \to O(S_X)$$

$$a \mapsto a^*|S_X$$

Exact sequence

$$1 \to N_X \to Aut(X) \to O(S_X)$$

with N_X finite, cyclic. Set

$$n_X = |N_X|.$$

$n_X|m_X$, hence $\varphi(n_X) \leqslant 20$.

Question. What are the possible values of n_X ?

$$\mathrm{rank}(T_X) = r_X \varphi(n_X).$$

Results for $r_X = 1$

Vorontsov (1982), Kondo (1992).

Machida-Oguiso (1998), Zhang (1998).

Book of Huybrechts (2016) defines n_X, m_X, summarizes results, raises questions.

Vorontsov, Kondo

Results for $r_X = 1$.

$$A = \{12,\ 28,\ 36,\ 42,\ 44,\ 66\}$$

$$B = \{3,\ 9,\ 27,\ 5,\ 25,\ 7,\ 11,\ 13,\ 17,\ 19\}.$$

Theorem. (Vorontsov, 1982, Kondo, 1992) Let $n \in \mathbf{Z}$ with $n \geqslant 3$. Then there exists a projective $K3$ surface X such that $n_X = n$ and $r_X = 1 \iff n \in A \cup B$.

Moreover,

$$n_X \in A \iff \text{the lattice } T_X \text{ is unimodular.}$$

and X is unique up to isomorphism.

Vorontsov, Kondo

Vorontsov states the result in the non-unimodular case (B), no proof. Theorem as above stated by Kondo, proofs by Kondo, Machida-Oguiso, Zhang.

$r_X = 1$

X a $K3$ surface with $n_X = n$ and $r_X = 1 \implies$

there exists $a \in Aut(X)$ such that

- $a^*|S_X$ is the identity, and
- the characteristic polynomial of $a^*|T_X$ is Φ_n.

$r_X = 1$

Theorem. Let $n \in \mathbf{Z}$ with $n \geqslant 3$ and $\varphi(n) \leqslant 20$; set

$$C = \Phi_n.$$

There exists a projective $K3$ surface X such that

$$n_X = n \text{ and } r_X = 1$$

if and only if

(i) $C(-1) = 1$.

(ii) If $C(1) = 1$, then $n \equiv 0 \pmod 2$ and $\deg(C) \equiv 4 \pmod 8$.

Moreover, X is unique up to isomorphism, and

$$C(1) = 1 \iff \text{the lattice } T_X \text{ is unimodular.}$$

EXAMPLE

p prime, $p \neq 2$,

$$C(X) = \Phi_p(X) = X^{p-1} + \cdots + X + 1$$

$$C(-1) = 1$$

$\Longrightarrow$ condition (i)

$$C(1) = p$$

$\Longrightarrow$ condition (ii).

KONDO, VORONTSOV

$$A = \{12,\ 28,\ 36,\ 42,\ 44,\ 66\}$$

$$B = \{3,\ 9,\ 27,\ 5,\ 25,\ 7,\ 11,\ 13,\ 17,\ 19\}.$$

$n \in \mathbf{Z}$ with $n \geqslant 3$ and that $\varphi(n) \leqslant 20$. Then

$$n \in B \iff \Phi_n(-1) = 1 \text{ and } \Phi_n(1) > 1.$$

$$n \in A \iff$$

$\Phi_n(1) = \Phi_n(-1) = 1$, n is even, and $\varphi(n) \equiv 4 \pmod 8$.

$r_X > 1$

Theorem. $n, r \in \mathbf{Z}$ with $n \geqslant 3$ and $r \geqslant 1$, and $r\varphi(n) \leqslant 20$, set

$$C = \Phi_n^r.$$

There exists a projective $K3$ surface X such that

$$n_X = n \text{ and } r_X = r$$

if and only if

(i) $C(-1)$ is a square.

(ii) If $C(1) = 1$, then $n \equiv 0 \pmod 2$ and $\deg(C) \equiv 4 \pmod 8$.

Moreover,

$$C(1) = 1 \implies \text{the lattice } T_X \text{ is unimodular.}$$

$$X \text{ is unique up to isomorphism} \iff r = 1.$$

$r > 1$

$$A = \{12,\ 28,\ 36,\ 42,\ 44,\ 66\}$$

$$B = \{3,\ 9,\ 27,\ 5,\ 25,\ 7,\ 11,\ 13,\ 17,\ 19\}.$$

$$C = \{4,\ 6,\ 8,\ 14,\ 16,\ 18,\ 22\}.$$

Theorem. $n \in \mathbf{Z}$ with $n \geqslant 2$ and $\varphi(n) \leqslant 20$. There exists a projective $K3$ surface X with $n_X = n \iff$

$$n \in A \cup B \cup C.$$

$n, r \in \mathbf{Z}$ with $n \geqslant 2$, $r \geqslant 1$ and $r\varphi(n) \leqslant 20$. There exists a projective $K3$ surface X with

$$n_X = n \text{ and } r_X = r$$

$\iff$

(i) $r = 1$ and $n \in A \cup B$.

(ii) $r = 2$ and $n \in C \cup \{3, 5, 7, 11\}$.

(iii) $r = 3$ and $n \in \{3, 5, 7, 9, 12\}$.

(iv) $r = 4$ and $n \in \{3, 4, 5, 8\}$.

(v) $r = 5$ and $n \in \{3, 5\}$.

(vi) $r = 6$ or 10 and $n \in \{3, 4, 6\}$.

(vii) $r = 8$ and $n \in \{3, 4\}$.

(viii) $r = 7$ or 9 and $n = 3$.

m_X

Question. What are the possiblities for m_X ?

Theorem. $m \in \mathbf{Z}$ with $m \geqslant 1$ and $\varphi(m) \leqslant 20$. Suppose that $m = 1$ or m is even. Then there exists a projective $K3$ surface X with

$$m_X = m.$$

m_X

Question. What are the possiblities for m_X ?

Theorem. $m \in \mathbf{Z}$ with $m \geqslant 1$ and $\varphi(m) \leqslant 20$. Suppose that $m = 1$ or m is even. Then there exists a projective $K3$ surface X with

$$m_X = m.$$

Question. What about m odd, $m > 1$?

OPEN QUESTIONS

Does there exist a $K3$ surface X with m odd, $m > 1$?

What are the possibilities for

$$(n_X, m_X) \text{ ?}$$

for

$$(n_X, m_X, r_X) \text{ ?}$$

HODGE ENDOMORPHISMS

Hodge structures

X complex $K3$ surface, $L_X = H^2(X, \mathbf{Z})$, Hodge structure

$$L_X \otimes_{\mathbf{Z}} \mathbf{C} = L^{2,0} \oplus L^{1,1} \oplus L^{2,0}$$

$L^{i,j} = H^{i,j}(X)$.

Transcendental lattice

$$T_X \subset L_X$$

induced Hodge structure

$$T_X \otimes_{\mathbf{Z}} \mathbf{C} = T^{2,0} \oplus T^{1,1} \oplus T^{2,0}$$

$T^{i,j} = L^{i,j} \cap (T_X \otimes_{\mathbf{Z}} \mathbf{C})$.

Hodge endomorphisms

$$\mathrm{End}_{Hdg}(T_X) = \{f : T_X \to T_X \text{ linear } | f(T^{i,j}) \subset T^{i,j}\}.$$

$$E_X = \mathrm{End}_{Hdg}(T_X) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Zarhin (1987) E_X is a number field.

$$alg(X) = 2$$

$\Longrightarrow$

E_X is totally real or CM.

$$alg(X) = 1$$

B. - van Geemen - Schütt (2025)

$$alg(X) = 1$$

$$\Longrightarrow$$

$$E_X = \mathbf{Q}.$$

SALEM FIELDS

Salem field

E of degree $\geqslant 4$, quadratic extension of a totally real field F.

Exactly one real place of F extends to (two) real places of E.

Example.

$$E = \mathbf{Q}(\lambda), \text{ where } \lambda \text{ is a Salem number.}$$

$alg(X) = 0$

B. - van Geemen - Schütt (2025)

$$alg(X) = 0$$

$\Longrightarrow$

E_X is totally real or a Salem field.

Salem fields and Salem numbers

B. - van Geemen - Schütt (2025)

Theorem. Let E be a Salem field. Then there exists a Salem number λ such that

$$E = \mathbf{Q}(\lambda).$$

Salem fields and Salem numbers

B. - van Geemen - Schütt (2025)

Theorem. Let E be a Salem field. Then there exists a Salem number λ such that

$$E = \mathbf{Q}(\lambda).$$

Two proofs :

- Proof of Chris Smyth.
- Proof based on work of Stark and Chinburg (1974, 1984).

STARK and CHINBURG

E a Salem field, totally real field F of degree d, E/F quadratic extension.

h_E, h_F class numbers, χ non-trivial character of $\mathrm{Gal}(E/F)$.

$L(s,\chi)$ Artin L-function.

$u = u_E = 1$ or 2.

Theorem. Set

$$\lambda_E = \exp(L'(0,\chi)(h_F/h_E)2^{d-1}u).$$

Then

$$\lambda_E \text{ is a Salem number, and } E = \mathbf{Q}(\lambda_E).$$

LEHMER

$$\lambda_{10} = 1,17628...$$

Question (Salem number question). Let λ be a Salem number. Is

$$\lambda \geqslant \lambda_{10}?$$

Question (Salem field question). Let E be a Salem field. Is

$$\lambda_E \geqslant \lambda_{10}?$$

Positive answer to the Salem number question $\implies$ positive answer to the Salem field question.

Positive answer to the Salem number question $\implies$ positive answer to the Salem field question.

Study of Salem fields and Salem numbers ?

Thank you

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