On the identification of elliptic curves that admit infinitely many twists satisfying the Birch--Swinnerton-Dyer conjecture

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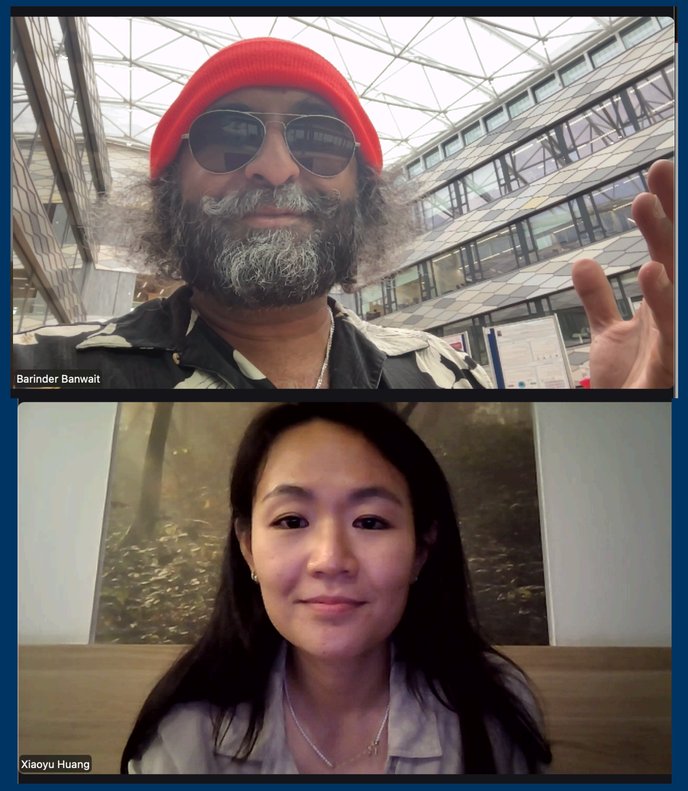

XVII ANTS taking over Rijksuniversiteit Groningen

8th July 2026

Peter Swinnerton-Dyer with Bryan Birch, July 2nd 1973, Solihull UK

Barinder S. Banwait, Coco Xiaoyu Huang (Temple)

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4. In this section we describe our results. We have applied the methods described in §§ 2, 3 to the curves

(30) $$\Gamma : y^2 = x^3 - Ax - B$$

for all integers A, B with $0 < |A| \leqq 20$, $0 < |B| \leqq 30$; to the curve

(31) $$y^2 = x^3 - D$$

for all integers D with $|D| \leqq 400$; and to the curve

(32) $$y^2 = x^3 - Dx$$

for all integers D with $|D| \leqq 200$. We have also dealt with a few higher ranges of A, B and a few larger values of D. The calculations needed for each curve take very little time; as we had to put a great deal of effort into producing a working program, it would have been uneconomical not to do a lot of cases.

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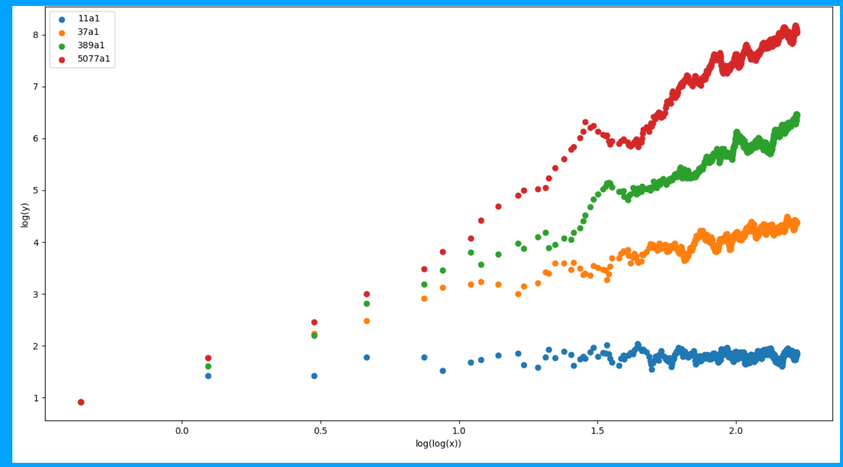

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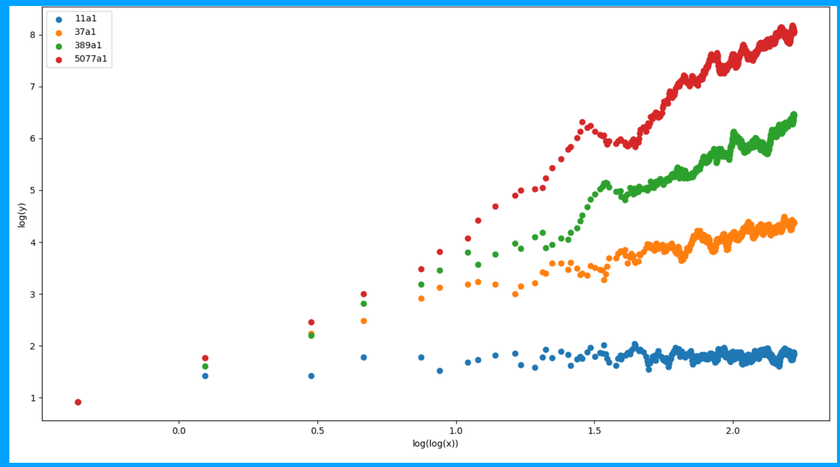

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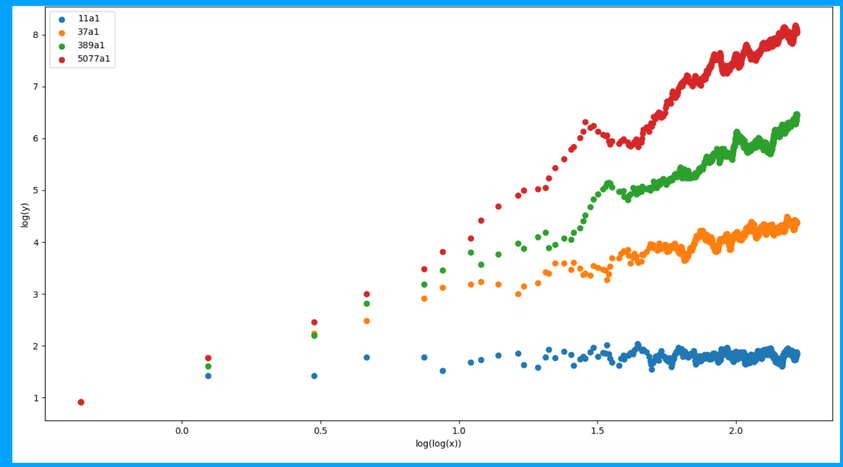

Selfie with BJB

"Our application for machine time was approved, with low priority—on certain nights we could use EDSAC from midnight until it broke down, which was typically after a couple of hours." — Bryan Birch

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$$a_p = \begin{cases} p+1-N_p & p \nmid N \quad \text{(good reduction)}, \\ 1 & p \mid N, \text{ split multiplicative reduction}, \\ -1 & p \mid N, \text{ non-split multiplicative reduction}, \\ 0 & p \mid N, \text{ additive reduction}. \end{cases}$$

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$$L(E,s) := \prod_{p|N} (1 - a_p p^{-s})^{-1} \cdot \prod_{p \nmid N} (1 - a_p p^{-s} + p^{1-2s})^{-1}$$

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(Weak) $\operatorname{ord}_{s=1} L(E,s) = r_E$

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(Weak) $\operatorname{ord}_{s=1} L(E,s) = r_E$

(Strong) $$\frac{L^{(r)}(E,1)}{r!} = \frac{\Omega(E/\mathbb{Q}) \cdot \mathrm{Reg}(E/\mathbb{Q}) \cdot |Ш(E/\mathbb{Q})| \cdot \prod_p c_p}{|E(\mathbb{Q})_{tors}|^2}.$$

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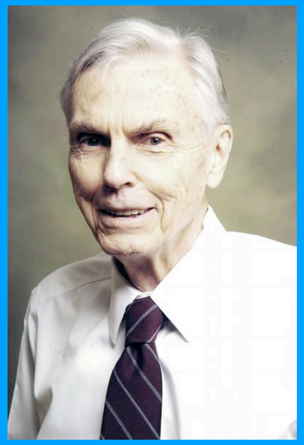

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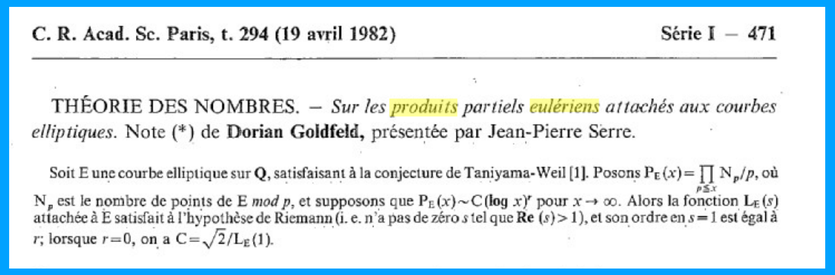
This remarkable conjecture relates the behavior of a function L at a point where it is not at present known to be defined to the order of a group Ш which is not known to be finite! — John Tate,1974



C. R. Acad. Sc. Paris, t. 294 (19 avril 1982) Série I – 471

THÉORIE DES NOMBRES. – *Sur les produits partiels eulériens attachés aux courbes elliptiques.* Note (*) de **Dorian Goldfeld,** présentée par Jean-Pierre Serre.

Soit E une courbe elliptique sur **Q**, satisfaisant à la conjecture de Taniyama-Weil [1]. Posons $P_E(x)=\prod_{p\leq x} N_p/p$, où N_p est le nombre de points de E *mod p*, et supposons que $P_E(x)\sim C(\log x)^r$ pour $x\to\infty$. Alors la fonction $L_E(s)$ attachée à E satisfait à l'hypothèse de Riemann (i. e. n'a pas de zéro *s* tel que **Re** $(s)>1$), et son ordre en $s=1$ est égal à *r*; lorsque $r=0$, on a $C=\sqrt{2}/L_E(1)$.

$$\prod_{p<x}\frac{N_p}{p}\ \to\ C(\log x)^r$$

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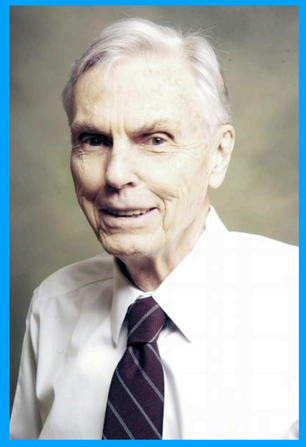

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What are the known cases of BSD?

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(7.3) **Theorem.** *Assume that* $L(E,1)=0$. *Then there is a rational point* P *in* $E(\mathbb{Q})$ *such that* $L'(E,1)=\alpha\cdot\Omega\cdot\langle P,P\rangle$ *with* $\alpha\in\mathbb{Q}^\times$. *In particular:*

1) *If* $L'(E,1)\neq 0$, *then* $E(\mathbb{Q})$ *contains elements of infinite order.*

2) *If* $L'(E,1)\neq 0$ *and* $\operatorname{rank} E(\mathbb{Q})=1$, *then formula* (7.2) *is true for some non-zero rational number* α.

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СЕРИЯ МАТЕМАТИЧЕСКАЯ Том 52, № 3, 1988

УДК 519.4

КОЛЫВАГИН В. А.

КОНЕЧНОСТЬ $E(\mathbf{Q})$ И $Ш(E,\mathbf{Q})$ ДЛЯ ПОДКЛАССА КРИВЫХ ВЕЙЛЯ

Введение

Пусть E — эллиптическая кривая, определенная над полем рациональных чисел $\mathbf{Q}$, $L(E,\mathbf{Q},s)=\sum_{n=1}^{\infty} a_n n^{-s}$, $a_n\in\mathbf{Z}$ — каноническая L-функция E над $\mathbf{Q}$. Гипотеза Берча — Суиннертона — Дайера утверждает, что ранг

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Theorem. Strong BSD is known if $r_{an} \leq 1$ and $N_E \leq 5{,}000$

Proceeds by verifying BSD(E, p)

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$$\frac{L^{(r)}(E,1)}{r!} = \frac{\Omega(E/\mathbb{Q}) \cdot \mathsf{Reg}(E/\mathbb{Q}) \cdot |Ш(E/\mathbb{Q})| \cdot \prod_p c_p}{|E(\mathbb{Q})_{tors}|^2}.$$

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Theorem 11.1. *Suppose E is as above and Ш is the Tate-Shafarevich group of E over K.*

(i) If $L(\bar{\psi}, 1) \neq 0$ then $E(K)$ is finite, Ш is finite, and for all primes $\mathfrak{p}$ of K not dividing $\#(\mathcal{O}^\times)$,

$$\#(Ш_{\mathfrak{p}^\infty}) = \mathbf{N}(\mathfrak{p})^{m(\mathfrak{p})} \text{ where } m(\mathfrak{p}) = \mathrm{ord}_{\mathfrak{p}}\left(\#(E(K))\frac{L(\bar{\psi}, 1)}{\Omega}\right).$$

In other words, Gross' refinement [9] of the Birch and Swinnerton-Dyer conjecture for E is true up to powers of primes dividing $\#(\mathcal{O}^\times)$.

The 'main conjectures' of Iwasawa theory for imaginary quadratic fields, 1991

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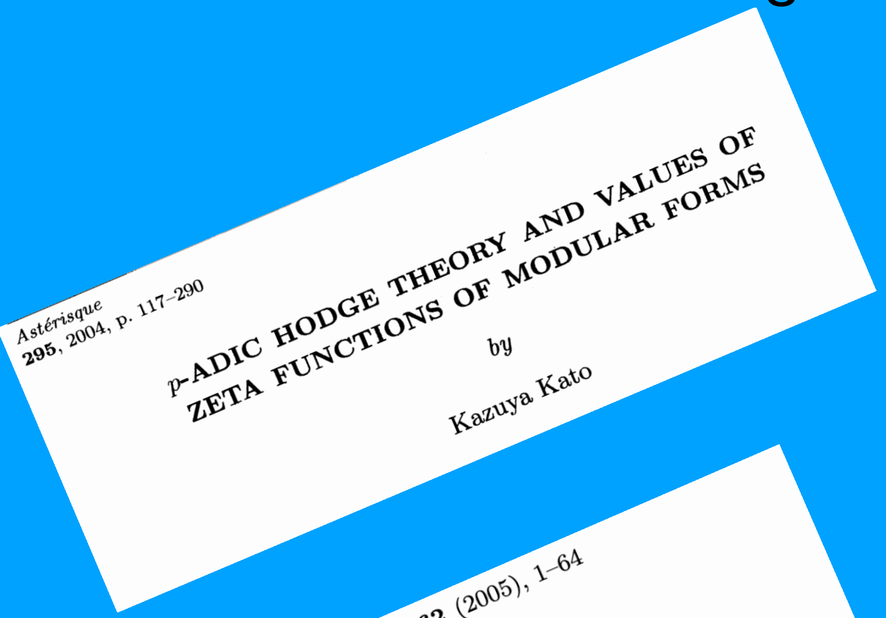

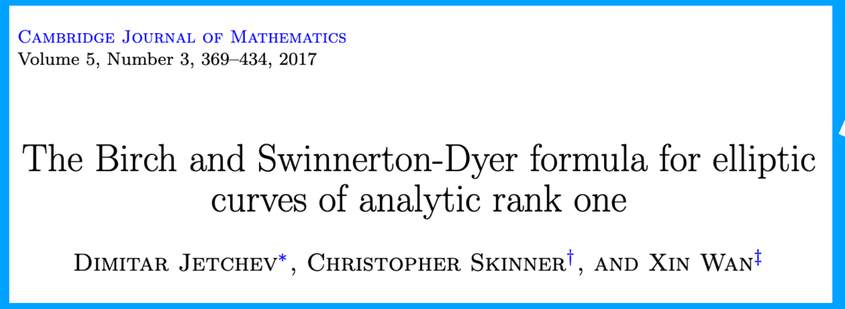

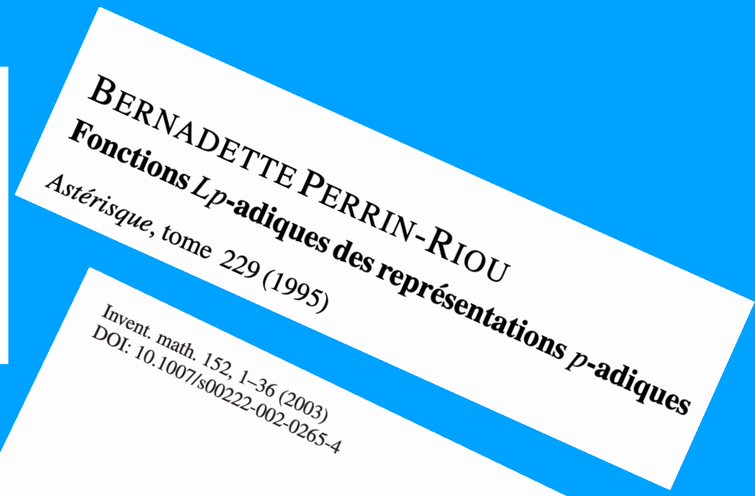

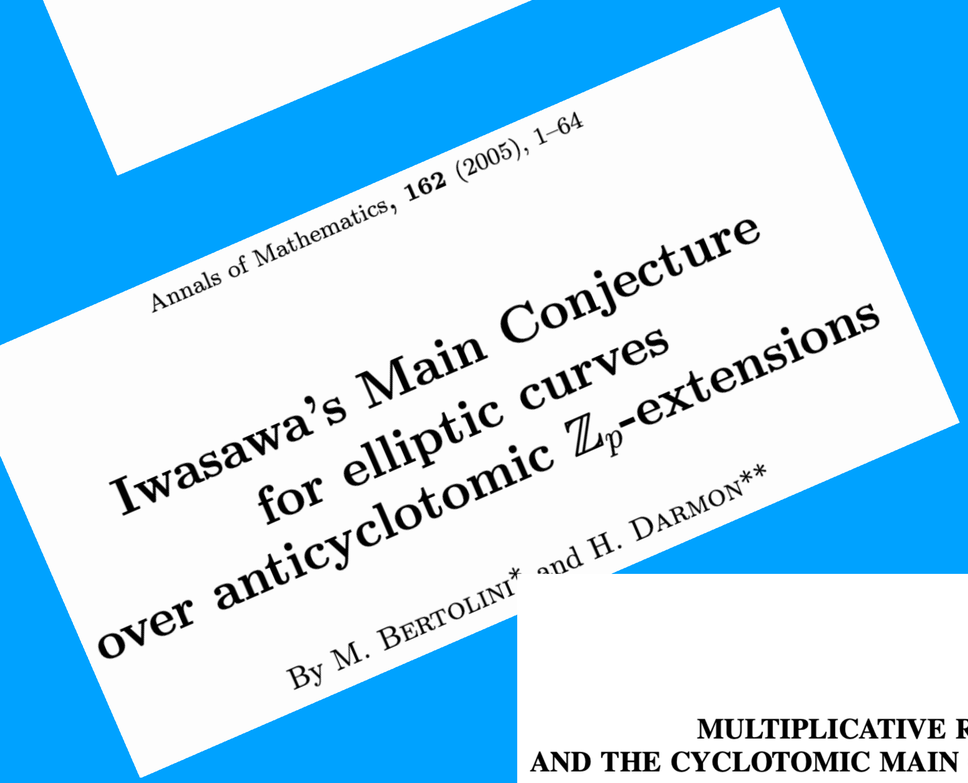

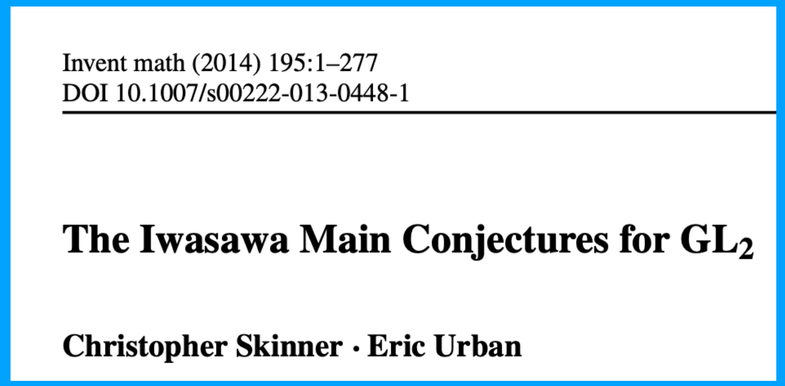

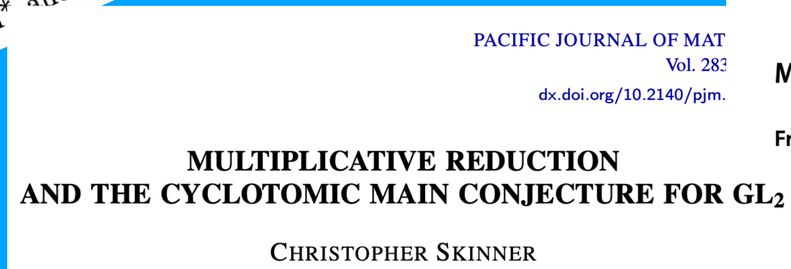

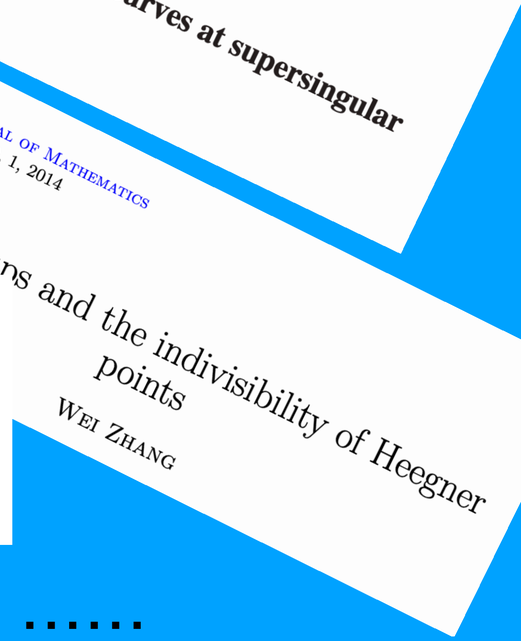

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[math.NT] 11 Sep 2024

ZETA ELEMENTS FOR ELLIPTIC CURVES AND APPLICATIONS

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doi:10.1112/plms/pdu059

Quadratic Twists of elliptic curves

John Coates, Yongxiong Li, Ye Tian and Shuai Zhai

For Bryan Birch and Peter Swinnerton-Dyer

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Theorem 1.7. *Let E be an elliptic curve of conductor N denoted by* 46a1, 62a1, 66b1, 69a1, 77c1, 94a1, 105a1, 106d1, 114b1, 115a1, 118c1, 118d1, 141b1, 141c1, 141e1 *or* 142c1 *in Cremona's labelling. Let $M > 1$ be a square-free integer with $(M, N) = 1$ and E^M the quadratic twist of E by the character associated to the extension $\mathbb{Q}(\sqrt{M})/\mathbb{Q}$. Suppose that the following conditions hold:*

(a) *$L(1, E^M) \neq 0$, and*
(b) *E has ordinary reduction at the primes dividing M.*

Then the BSD conjecture holds for E^M, i.e. $E^M(\mathbb{Q})$ and $Ш(E^M)$ are finite, and

$$\frac{L(1, E^M)}{\Omega_{E^M}} = \frac{\#Ш(E^M) \cdot \prod_{\ell \nmid \infty} c_\ell(E^M)}{\#E^M(\mathbb{Q})^2_{\mathrm{tor}}}.$$

Moreover, the conditions (a) and (b) are satisfied by infinitely many M.

10.4.3. *The Birch and Swinnerton-Dyer formula: a rank zero quadratic twist family.*

Theorem 10.12. *Let E be a semistable elliptic curve defined over $\mathbb{Q}$ with conductor N, and $M > 1$ a square-free integer with $(M, N) = 1$. Let $E^{(M)}$ denote the quadratic twist of E by the character associated to the quadratic extension $\mathbb{Q}(\sqrt{M})/\mathbb{Q}$. Suppose that the following conditions hold.*

(i) *We have*
$$L(1, E^{(M)}) \neq 0,$$
(ii) *The 2-part of the BSD formula holds for $E^{(M)}$,*
(iii) $a_3(E) = 0$,
(iv) *For all odd primes p, $E[p]$ is an absolutely irreducible $G_{\mathbb{Q}}$-representation,*
(v) *For any prime $p|N$, there exists a multiplicative prime $q \neq p$ at which $E[p]$ is ramified,*
(vi) *E has ordinary reduction at prime divisors of M.*

Then the Birch and Swinnerton-Dyer conjecture is true for $E^{(M)}$, that is, $E^{(M)}(\mathbb{Q})$ and $Ш(E^{(M)})$ are finite, and
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Example 1. For semistable elliptic curves E up to conductor 150, the conditions of Theorem 10.12 are satisfied by[22] the curves denoted in the LMFDB database by

- 46a1, 69a1, 77c1, 94a1, 114b1, 141b1 and 142c1,
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for infinitely many M. Here for the elliptic curves in the first bullet point we rely on [41, Thm. 1.5] for the conditions (i)-(ii), and for the curves in the second bullet point on [148, Thm. 1]. This gives the first infinite families of non-CM elliptic curves satisfying the Birch and Swinnerton-Dyer conjecture.

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THEOREM 1.5. *Let E and M be as in Theorem 1.1. Assume further that*

(1) *$Ш(E')[2] = 0$;*
(2) *all primes ℓ which divide $2C$ split in $\mathbb{Q}(\sqrt{M})$;*
(3) *the 2-part of the Birch and Swinnerton-Dyer conjecture holds for E.*

Then the 2-primary component of $Ш(E^{(M)})$ is zero, and the 2-part of the Birch and Swinnerton-Dyer conjecture holds for $E^{(M)}$.

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Theorem 10.12. *Let E be a semistable elliptic curve defined over $\mathbb{Q}$ with conductor N, and $M > 1$ a square-free integer with $(M, N) = 1$. Let $E^{(M)}$ denote the quadratic twist of E by the character associated to the quadratic extension $\mathbb{Q}(\sqrt{M})/\mathbb{Q}$. Suppose that the following conditions hold.*

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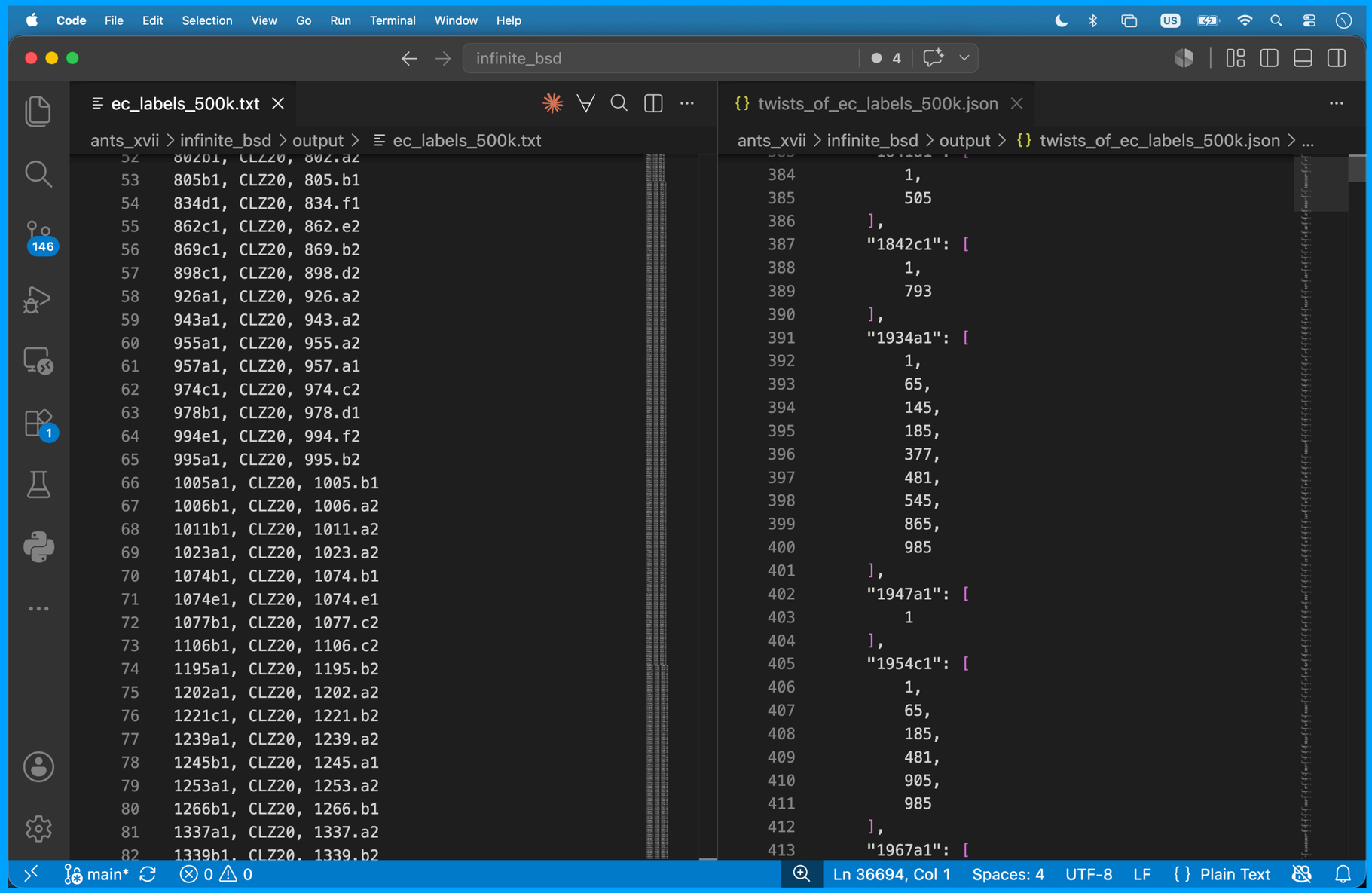
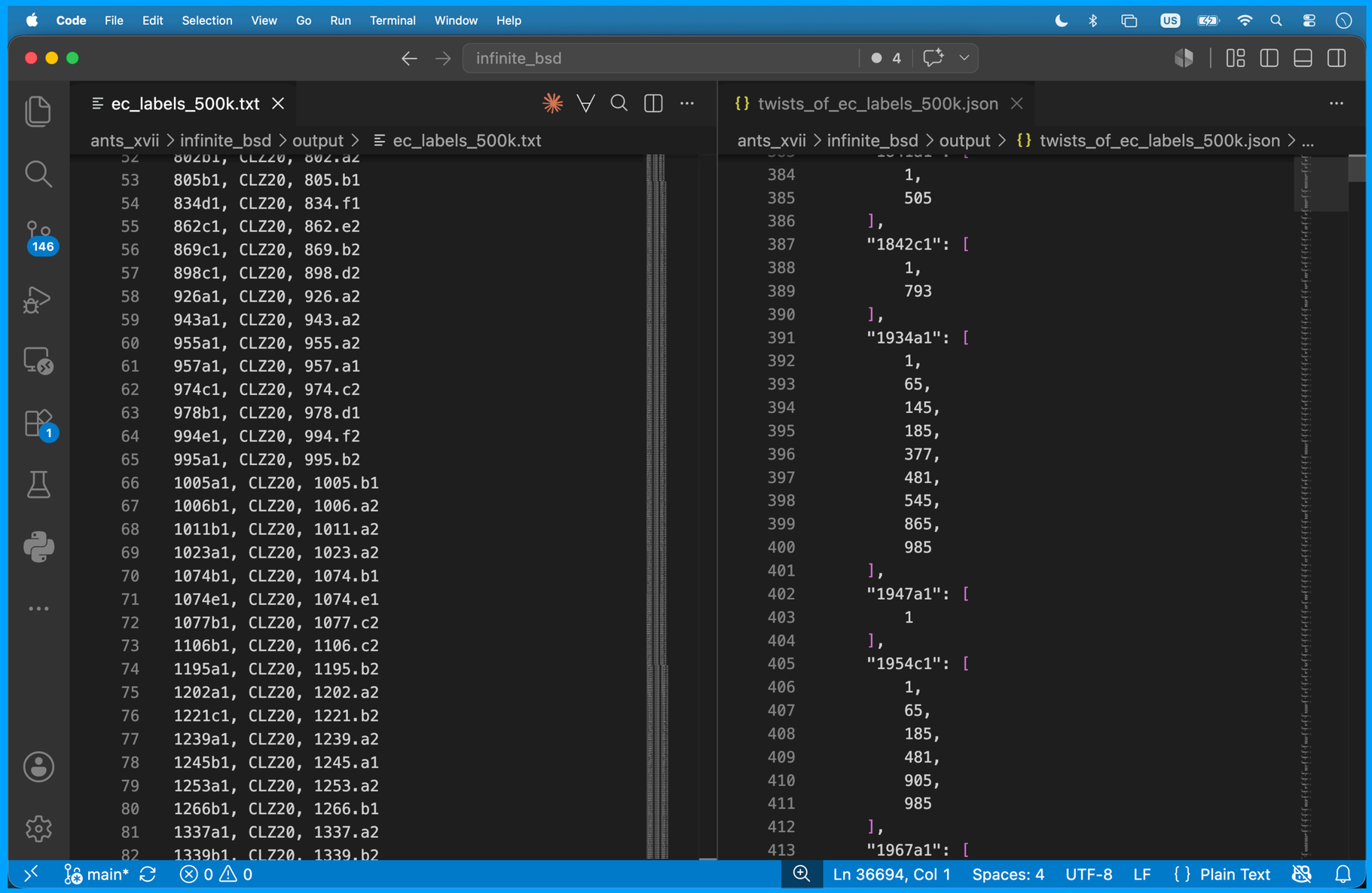
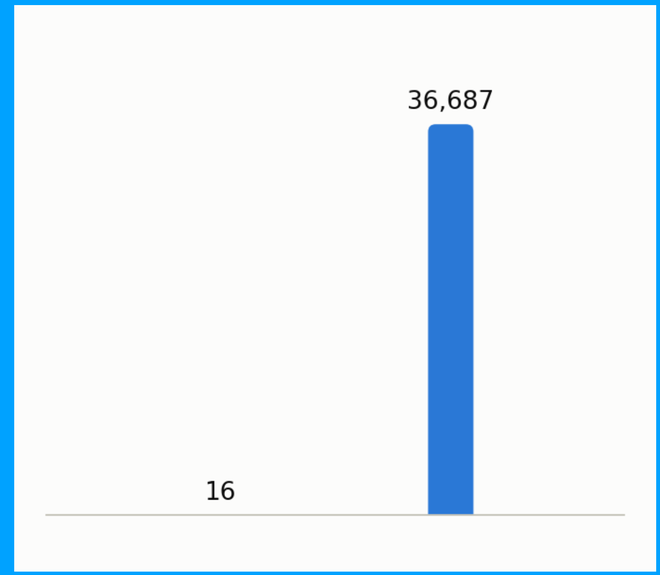
Therefore, by Theorem 2.13 and Theorem 2.14, such E as above admit infinitely many quadratic twists E_d that unconditionally satisfy BSD.



https://github.com/cocoxhuang/ants_xvii/tree/main/ants_xvii/infinite_bsd





~2,300x more curves 😎

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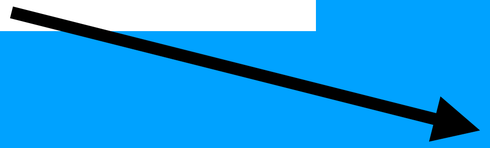

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Lemma 4.1. *Let* $\ell \nmid N$ *be a prime. Then the following are equivalent:*

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3) ℓ *is inert in both* $\mathbb{Q}(E[2])$ *and* $\mathbb{Q}(E_0[2])$.

Kriz, Li, 2019

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straightforward, and combining this with the Keating-Snaith conjecture for $L\left(\frac{1}{2}, E_d\right)$, we are led to formulate the following conjecture.

Conjecture 1 *Let E be given by the model $y^2 = f(x)$ for a monic cubic polynomial f with integer coefficients. Let K denote the splitting field of f over $\mathbb{Q}$, and let $G = Gal(K/\mathbb{Q})$ denote the associated Galois group, which we view as a subgroup of S_3. Given $g \in G$ let $c(g)$ denote one plus the number of fixed points of g (viewed as an element of S_3), and define*

$$\mu(E) = -\frac{1}{2} - \frac{1}{|G|}\sum_{g\in G} \log c(g) \text{ and } \sigma(E) = 1 + \frac{1}{|G|}\sum_{g\in G} (\log c(g))^2 .$$

As d ranges over $\mathcal{E}$, the distribution of $\log\left(|Ш(E_d)|/\sqrt{|d|}\right)$ is approximately Gaussian with mean $\mu(E)\log\log|d|$ and variance $\sigma(E)^2\log\log|d|$. More precisely, for any fixed $V \in \mathbb{R}$ and as $X \to \infty$,

$$\left|\left\{d \in \mathcal{E}, 20 < |d| \le X : \frac{\log\left(|Ш(E_d)|/\sqrt{|d|}\right) - \mu(E)\log\log|d|}{\sqrt{\sigma(E)^2\log\log|d|}} \ge V\right\}\right|$$

is

$$\sim |\{d \in \mathcal{E}, |d| \le X\}\left(\frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-\frac{x^2}{2}}dx\right).$$

How did we end up doing this?

What we did (2/2)

We were actually attempting to verify a conjecture of Radziwill—Soundararajan:

straightforward, and combining this with the Keating-Snaith conjecture for $L\left(\frac{1}{2}, E_d\right)$, we are led to formulate the following conjecture.

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https://radzi-sound-conjecture.netlify.app/46a1_maxd100000_nf100_20260116_185105_pdf

Where do we go from here?

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LeanBridge: Issuing formally verified certificates showing full BSD

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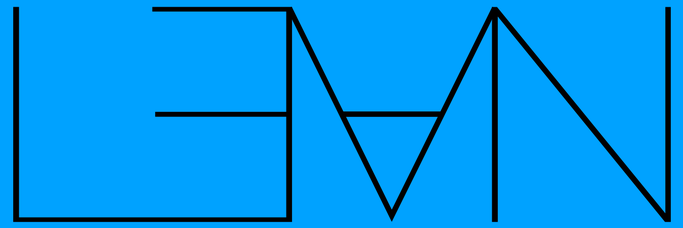

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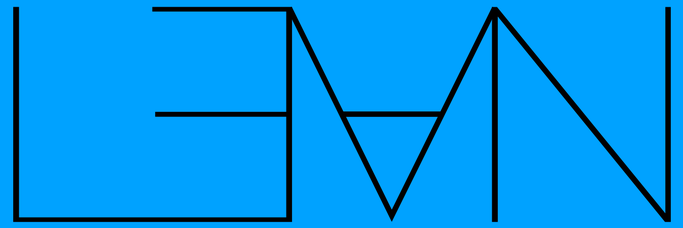

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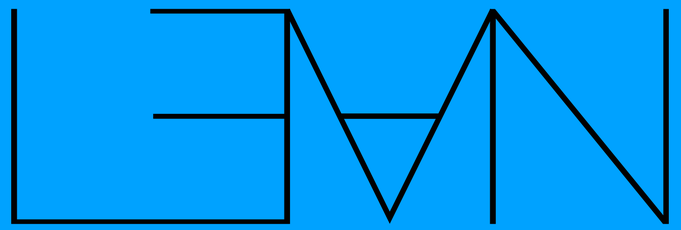

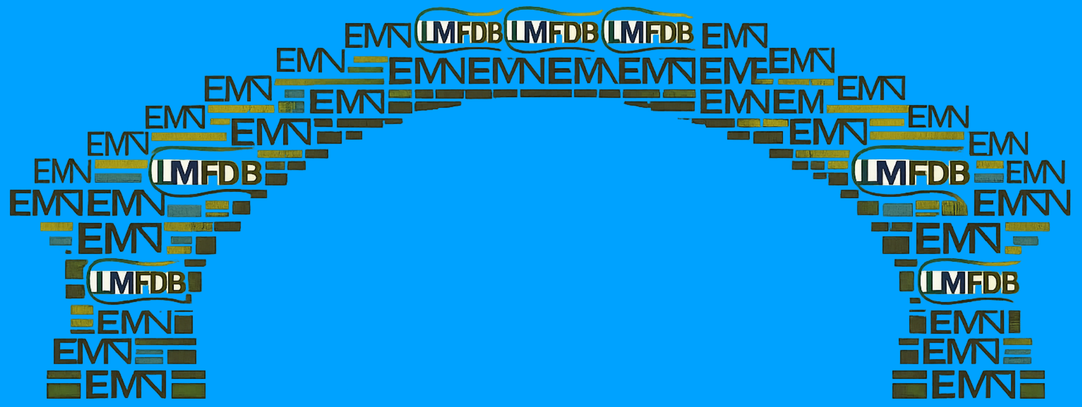

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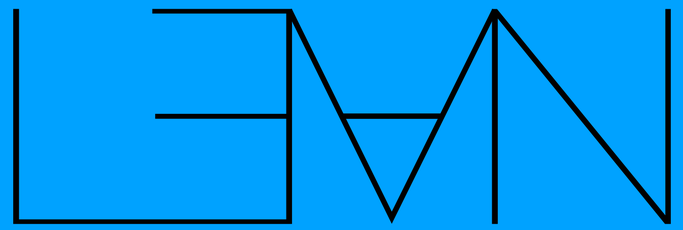

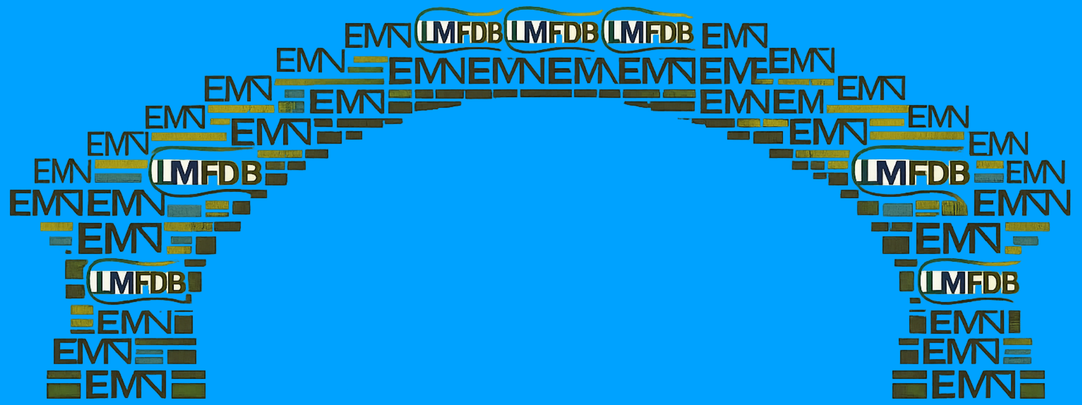

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