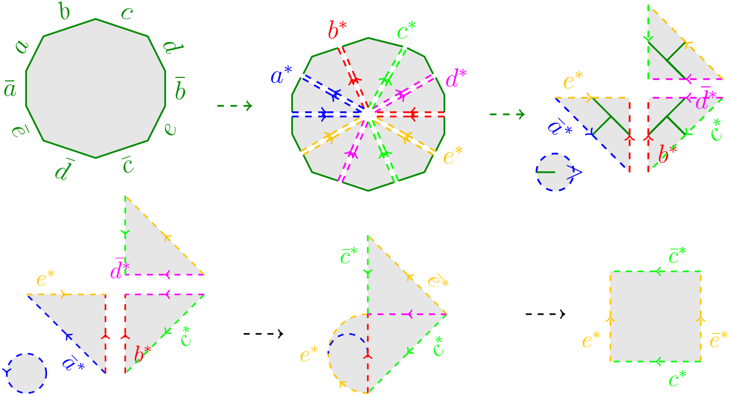


Computing the cohomology of Shimura curves in quasi-linear time.

Rayane Baït - Aurel Page

Introduction

Elliptic modular forms and the homology of modular curves.

From Eichler–Shimura

$$H_1(X_0(N), \mathbb{C}; cusps) \simeq S_2(N) \oplus \overline{S_2(N)}$$

In [Man73], Yu. Manin shows how to compute the Hecke-module

$$H_1(X_0(N), \mathbb{C}; cusps)$$

with modular symbols.

MOTTO

To compute elliptic modular forms, compute the homology of a modular curve.

Introduction

Elliptic modular forms and the homology of modular curves.

From Eichler–Shimura

$$H_1(X_0(N), \mathbb{C}; cusps) \simeq S_2(N) \oplus \overline{S_2(N)}$$

In [Man73], Yu. Manin shows how to compute the Hecke-module

$$H_1(X_0(N), \mathbb{C}; cusps)$$

with modular symbols.

MOTTO

To compute elliptic modular forms, compute the homology of a modular curve.

Introduction

Elliptic modular forms and the homology of modular curves.

From Eichler–Shimura

$$H_1(X_0(N), \mathbb{C}; cusps) \simeq S_2(N) \oplus \overline{S_2(N)}$$

In [Man73], Yu. Manin shows how to compute the Hecke-module

$$H_1(X_0(N), \mathbb{C}; cusps)$$

with modular symbols.

MOTTO

To compute elliptic modular forms, compute the homology of a modular curve.

Introduction

Hilbert modular forms and the cohomology of Shimura curves.

From Eichler–Shimura and Jacquet–Langlands

$$S_k(\eta) \hookrightarrow H^1(\Gamma_\eta, \mathrm{Sym}_{k-2}(\mathbb{C}^2)).$$

In [GV11], Greenberg and Voight show how to compute the Hecke-module

$$H^1(\Gamma_\eta, \mathrm{Sym}_{k-2}(\mathbb{C}^2)).$$

MOTTO

To compute Hilbert modular forms, compute the cohomology of a Shimura curve.

Introduction

Hilbert modular forms and the cohomology of Shimura curves.

From Eichler–Shimura and Jacquet–Langlands

$$S_k(\eta) \hookrightarrow H^1(\Gamma_\eta, \mathrm{Sym}_{k-2}(\mathbb{C}^2)).$$

In [GV11], Greenberg and Voight show how to compute the Hecke-module

$$H^1(\Gamma_\eta, \mathrm{Sym}_{k-2}(\mathbb{C}^2)).$$

MOTTO

To compute Hilbert modular forms, compute the cohomology of a Shimura curve.

Some notations:

- Upper half plane $\mathfrak{h}$ $\qquad \mathfrak{h} = \{\tau \in \mathbb{C} | 0 < Im(\tau)\}$.
- Projective special linear group $PSL_2(\mathbb{R})$ $\qquad$ Acts on $\mathfrak{h}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}$

  $\rightarrow$ orientation preserving isometries.
- Fuchsian group Γ $\qquad$ Discrete subgroup of $PSL_2(\mathbb{R})$.
- Γ is co-compact $\qquad$ $\Gamma \backslash \mathfrak{h}$ is compact.

The quotient $X_\Gamma = \Gamma \backslash \mathfrak{h}$ is a complex 1-orbifold. In particular, a compact orientable topological surface.

- genus of Γ $\qquad$ genus of $\Gamma \backslash \mathfrak{h}$
- elliptic elements of Γ $\qquad$ conjugacy classes of maximal finite cyclic subgroups with orders $(\nu_1, \dots, \nu_f)$
- signature of Γ $\qquad$ $(g, \nu_1, \dots, \nu_f)$: uniquely determines Γ up to isomorphism.
- co-area μ of Γ $\qquad$ Area of $\Gamma \backslash \mathfrak{h}$.

Some notations:

- Upper half plane $\mathfrak{h}$ — $\mathfrak{h} = \{\tau \in \mathbb{C} | 0 < Im(\tau)\}$.
- Projective special linear group $PSL_2(\mathbb{R})$ — Acts on $\mathfrak{h}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}$
  $\rightarrow$ orientation preserving isometries.
- Fuchsian group Γ — Discrete subgroup of $PSL_2(\mathbb{R})$.
- Γ is co-compact — $\Gamma \backslash \mathfrak{h}$ is compact.

The quotient $X_\Gamma = \Gamma \backslash \mathfrak{h}$ is a complex 1-orbifold. In particular, a compact orientable topological surface.

- genus of Γ — genus of $\Gamma \backslash \mathfrak{h}$
- elliptic elements of Γ — conjugacy classes of maximal finite cyclic subgroups with orders $(\nu_1, \dots, \nu_f)$
- signature of Γ — $(g, \nu_1, \dots, \nu_f)$: uniquely determines Γ up to isomorphism.
- co-area μ of Γ — Area of $\Gamma \backslash \mathfrak{h}$.

In this talk

Problem :

Given a weight $2 \leq k$ and a co-compact Fuchsian group $\Gamma \subset PSL_2(\mathbb{R})$, efficiently compute $H^1(\Gamma, W_k(\mathbb{C}))$ where $W_k(\mathbb{C}) = Sym_{k-2}(\mathbb{C}^2)$.

Initial idea:

Compute any presentation of Γ and solve a linear system $\rightarrow \tilde{O}(\mu^3 k^3)$ with naive linear algebra.

Our approach :

Compute a better presentation of Γ using the geometry of $\Gamma \backslash \mathfrak{h}$ and solve a structured linear system $\rightarrow \tilde{O}(\mu k^3)$.

In this talk

Problem :

Given a weight $2 \leq k$ and a co-compact Fuchsian group $\Gamma \subset PSL_2(\mathbb{R})$, efficiently compute $H^1(\Gamma, W_k(\mathbb{C}))$ where $W_k(\mathbb{C}) = Sym_{k-2}(\mathbb{C}^2)$.

Initial idea:

Compute any presentation of Γ and solve a linear system $\rightarrow \tilde{O}(\mu^3 k^3)$ with naive linear algebra.

Our approach :

Compute a better presentation of Γ using the geometry of $\Gamma \backslash \mathfrak{h}$ and solve a structured linear system $\rightarrow \tilde{O}(\mu k^3)$.

We want to compute good presentations for co-compact Fuchsian groups Γ: interpret Γ as $\pi_1^{orb}(\Gamma\backslash\mathfrak{h})$.

If $Y \subset \mathfrak{h}$ is the set of points with non-trivial stabilizer by Γ, then there is a short exact sequence

$$1 \to \pi_1(\mathfrak{h} - Y, v) \to \pi_1(\Gamma \backslash (\mathfrak{h} - Y), \Gamma.v) \to \pi_1^{orb}(\Gamma \backslash \mathfrak{h}) = \Gamma \to 1$$

and a surjective map $\pi_1(\Gamma \backslash (\mathfrak{h} - Y), \Gamma.v) \to \pi_1(\Gamma \backslash \mathfrak{h}, \Gamma.v)$.

Idea

Use the well-known compact orientable surface structure on $\Gamma \backslash \mathfrak{h}$ to compute $\pi_1(\Gamma \backslash \mathfrak{h})$ and push it to Γ.

Theorem :

Let Γ be a co-compact Fuchsian group of signature $(g, \nu_1, \ldots, \nu_f)$. Then Γ has presentations of the form :

- $\langle e_1, \ldots, e_{2g}, \gamma_1, \ldots, \gamma_f \mid \prod_{i=1}^{4g} e_{\phi(i)}^{\pm 1} \prod_{j=1}^{f} \gamma_j = 1, \gamma_1^{\nu_1} = \ldots = \gamma_f^{\nu_f} = 1 \rangle$
- $\langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_f \mid \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{f} \gamma_j = 1,\ \gamma_1^{\nu_1} = \ldots = \gamma_f^{\nu_f} = 1 \rangle$

respectively called **one word** *and* **geometric** *presentations of Γ.*

Application to cohomology

Definitions

Γ a group, M a Γ-module.

1-cocycles

A 1-cocycle $\sigma\colon \Gamma \to M$ is a map satisfying

$$\forall g, h \in \Gamma \quad \sigma(gh) = \sigma(g) + g.\sigma(h)$$

Denote $Z^1(\Gamma, M) = \{1-\text{cocycles}\colon \Gamma \to M\}$.

1-coboundaries

A 1-coboundary $\sigma\colon \Gamma \to M$ is a 1-cocycle of the form

$$h \mapsto h.v - v$$

for $v \in M$. Denote $B^1(\Gamma, M) = \{1-\text{coboundaries}\colon \Gamma \to M\}$.

The first cohomology group of Γ with coefficients in M:

$$H^1(\Gamma, M) = Z^1(\Gamma, M)/B^1(\Gamma, M)$$

Application to cohomology

Solving the weight 2 case.

For Hilbert modular forms, want to compute

$$H^1(\Gamma_\eta, \mathrm{Sym}_{k-2}(\mathbb{C}^2))$$

for some co-compact Fuchsian groups Γ_η. When $k = 2$, $\mathrm{Sym}_{k-2}(\mathbb{C}^2) = \mathbb{C}$ is trivial as Γ_η-module.

Application to cohomology

Back to the weight 2 case.

Assume given a one-word presentation

$$\Gamma \simeq \langle e_1, \ldots, e_{2g}, \gamma_1, \ldots, \gamma_f \mid \prod_{i=1}^{4g} e_{\phi(i)}^{\pm 1} \prod_{j=1}^{f} \gamma_j = 1,\ \gamma_1^{\nu_1} = \ldots = \gamma_f^{\nu_f} = 1 \rangle.$$

Each e_i appears exactly twice, once positively and once negatively:

$$\rightarrow \quad \Gamma^{ab} \simeq \mathbb{Z}^{2g} \oplus \left(\prod_{i=1}^{f} \mathbb{Z}/\nu_i\mathbb{Z} \right) / (1, \ldots, 1).\mathbb{Z}$$

Application to cohomology

Endow $\mathbb{C}$ with the trivial Γ-action. We have

$$H^1(\Gamma,\mathbb{C}) \simeq Hom(\Gamma,\mathbb{C}) \simeq (\Gamma^{ab})^* \simeq (\mathbb{Z}^{2g} \oplus \left(\prod_{j=1}^{f} \mathbb{Z}/\nu_j\mathbb{Z}\right)/(1,\ldots,1).\mathbb{Z})^*$$

and using the dual basis we get $H^1(\Gamma,\mathbb{C}) \simeq \mathbb{C}^{2g}$.

Application to cohomology

Hint at the weight k case.

Again: want to compute $H^1(\Gamma_\eta, \mathrm{Sym}_{k-2}(\mathbb{C}^2))$ for some co-compact Fuchsian groups Γ_η.

For $k > 2$, one word presentation not structured enough, geometric presentation too expensive: use 1-handle presentation.

Assume given a 1-handle presentation

$$\Gamma \simeq \langle a, b, e_3, \ldots, e_{2g}, \gamma_1, \ldots, \gamma_f \mid [a, b] \prod_{i=5}^{4g} e_{\phi(i)}^{\pm 1} \prod_{j=1}^{f} \gamma_j = 1,\ \gamma_1^{\nu_1} = \ldots = \gamma_f^{\nu_f} = 1 \rangle$$

$a^{\pm 1}$ and $b^{\pm 1}$ appear only in $[a, b]$.

Put $M = \mathrm{Sym}_{k-2}(\mathbb{C}^2)$. Two steps:

- Evaluate cocycles on generators, get

$$Z^1(\Gamma, M) \hookrightarrow M^{2g+f}$$

- Evaluate cocycles on relation $w = [a,b]w'$:

$$\sigma(w) = 0 \quad \implies \quad \sigma([a,b]) = -[a,b]\sigma(w')$$

  $\rightarrow$ enough structure for quasi-linear time!

- Uses a crucial lemma specific to $M = \mathrm{Sym}_{k-2}(\mathbb{C}^2)$.

Remainder of the talk

- Input: representing Γ.
- Output: representing elements of Γ (straight line programs).
- Main algorithm.
- Timing comparison.
- Work in progress.

If time permits (it won't):

- Finding the special presentations for Γ using algebraic topology.

Remainder of the talk

- Input: representing Γ.
- Output: representing elements of Γ (straight line programs).
- Main algorithm.
- Timing comparison.
- Work in progress.

If time permits (it won't):

- Finding the special presentations for Γ using algebraic topology.

Remainder of the talk

- Input: representing Γ.
- Output: representing elements of Γ (straight line programs).
- Main algorithm.
- Timing comparison.
- Work in progress.

If time permits (it won't):

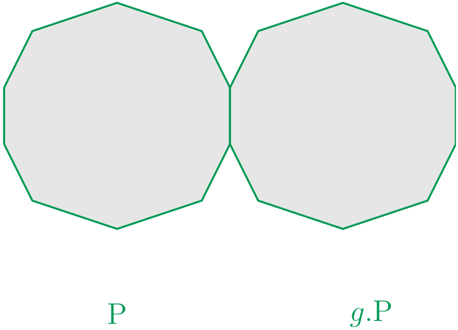
- Finding the special presentations for Γ using algebraic topology.

Input: representing Γ

Fundamental polygon P: polygon $P \subset \mathfrak{h}$ satisfying

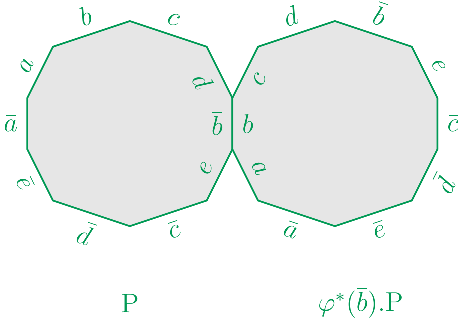
- $\sqcup_{g \in \Gamma} g.P = \mathfrak{h}$
- For $g \in \Gamma$, $g \neq 1 \implies g.\mathring{P} \cap \mathring{P} = \emptyset$

Denote by E the edges of P with the convention that $g.e \neq e$ for any $(e, g) \in E \times \Gamma$.



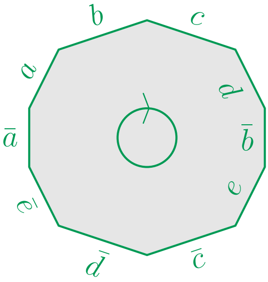
Side pairing (σ_1, φ^*): for each $e \in E$,

- $\varphi^*(e) =$ unique $g \in \Gamma$ with $gP \cap P = e$ → **monodromy** map: $E \to \Gamma$.
- $\sigma_1(e) =$ the edge $\varphi^*(e)^{-1}.e$ → **gluing** involution. No fixed-point by construction.



Here, $\sigma_1(e) = \bar{e}$ for every $e \in E(P)$.

Face permutation σ_2: Orientation of $P \rightarrow$ cyclic ordering of $E(P)$.



Denote corresponding face permutation $(abcd\bar{b}e\bar{c}\bar{d}\bar{e}\bar{a})$ by σ_2.

Input: representing Γ

Represent Γ by $(\sigma_1, \sigma_2, \varphi^*)$. Almost determines Γ abstractly:

Theorem: [Poincaré]

The set $\varphi^*(E)$ generates Γ and if $\sigma_1\sigma_2^{-1}$ has cycles $\{c_i\}_{i=1,\ldots,f}$ then a a cycle c_i is either a relation for Γ or an elliptic element of Γ so that $\{c_i^{\nu_i}\}$ is a complete set of relations where ν_i is the order of c_i in Γ.

Summary

Given $(\sigma_1, \sigma_2, \varphi^*)$:

- $\varphi^*(E) =$ generators for Γ.
- Cycles of $\sigma_1\sigma_2^{-1} =$ relations for Γ.

Output: representing elements of Γ

Straight line programs

Let $\Gamma = \langle g_1, g_2, g_3 \rangle$ a group. Want to represent $g = (g_1 g_2)^8 g_3 g_2$ and $h = g_3^2 (g_1 g_2)^8 g_2$. Two ways:

- Words in the generators:

$$(g, h) = (g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_3g_2,\\ g_3g_3g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_2)$$

- Straight line program:

1	2	3	4	5	6	7	8	9	10
g_1	g_2	g_3	$v_1 \times v_2$	v_4^8	$v_3 \times v_2$	$v_5 \times v_6$	v_3^2	$v_5 \times v_2$	$v_8 \times v_9$

evaluates to

$$[g_1,\ g_2,\ g_3,\ g_1g_2,\ (g_1g_2)^8,\ g_3g_2,\ (g_1g_2)^8 g_3g_2,\ g_3^2,\ (g_1g_2)^8 g_2,\ g_3^2(g_1g_2)^8 g_2]$$

Represent (g, h) as straight line program v + pointers to g and h in v.

Output: representing elements of Γ

Straight line programs

Let $\Gamma = \langle g_1, g_2, g_3 \rangle$ a group. Want to represent $g = (g_1 g_2)^8 g_3 g_2$ and $h = g_3^2 (g_1 g_2)^8 g_2$. Two ways:

- Words in the generators:

$$(g, h) = (g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_3g_2,\\ g_3g_3g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_2)$$

- Straight line program:

1	2	3	4	5	6	7	8	9	10
g_1	g_2	g_3	$v_1 \times v_2$	v_4^8	$v_3 \times v_2$	$v_5 \times v_6$	v_3^2	$v_5 \times v_2$	$v_8 \times v_9$

evaluates to

$$[g_1,\ g_2,\ g_3,\ g_1g_2,\ (g_1g_2)^8,\ g_3g_2,\ (g_1g_2)^8g_3g_2,\ g_3^2,\ (g_1g_2)^8g_2,\ g_3^2(g_1g_2)^8g_2]$$

Represent (g, h) as straight line program v + pointers to g and h in v.

Output: representing elements of Γ

Straight line programs

Let $\Gamma = \langle g_1, g_2, g_3 \rangle$ a group. Want to represent $g = (g_1 g_2)^8 g_3 g_2$ and $h = g_3^2 (g_1 g_2)^8 g_2$. Two ways:

- Words in the generators:

$$(g, h) = (g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_3g_2,\\ g_3g_3g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_1g_2g_2)$$

- Straight line program:

1	2	3	4	5	6	7	8	9	10
g_1	g_2	g_3	$v_1 \times v_2$	v_4^8	$v_3 \times v_2$	$v_5 \times v_6$	v_3^2	$v_5 \times v_2$	$v_8 \times v_9$

evaluates to

$$[g_1,\ g_2,\ g_3,\ g_1g_2,\ (g_1g_2)^8,\ g_3g_2,\ (g_1g_2)^8 g_3 g_2,\ g_3^2,\ (g_1g_2)^8 g_2,\ g_3^2 (g_1g_2)^8 g_2]$$

Represent (g, h) as straight line program v + pointers to g and h in v.

Output: representing elements of Γ

Straight line programs

Why ?

- Can represent many words in quasi-linear space instead of quadratic space.
- Can represent redundant words in quasi-linear space instead of exponential space.
- Can evaluate in any group(oid).
- 1-cocycles can be efficiently evaluated.

Main algorithm

Γ a co-compact Fuchsian group of co-area μ and signature $(g, \nu_1, \ldots, \nu_f)$.

Theorem: [B.–Page]

There exists an algorithm that given a side pairing (σ_1, φ^*) and a face permutation σ_2 for Γ associated with a convex fundamental polygon P, outputs a straight line program expressing one-word and $O(1)$-handles presentation of Γ in time $\tilde{O}(\mu + \sum_{i=1}^{f} \log(\nu_i))$.

Can also compute a geometric presentation in time $\tilde{O}(g\mu + \sum_{i=1}^{f} \log(\nu_i))$.

Corollary [B.–Page]

Let F be a totally real field of odd degree, $\eta \subset \mathbb{Z}_F$ a level, $k \geq 2$ a weight and g the genus of $\Gamma_\eta \backslash \mathfrak{h}$. There exists an algorithm that, given a side pairing (σ_1, φ^*) and a face permutation σ_2 associated with a convex fundamental polygon for Γ_η, computes a basis of $H^1(\Gamma_\eta, W_k(\mathbb{C}))$ as a straight line program in time $\tilde{O}(gk^3)$.

Main algorithm

Γ a co-compact Fuchsian group of co-area μ and signature $(g, \nu_1, \dots, \nu_f)$.

Theorem: [B.–Page]

There exists an algorithm that given a side pairing (σ_1, φ^*) and a face permutation σ_2 for Γ associated with a convex fundamental polygon P, outputs a straight line program expressing one-word and $O(1)$-handles presentation of Γ in time $\tilde{O}(\mu + \sum_{i=1}^{f} \log(\nu_i))$.

Can also compute a geometric presentation in time $\tilde{O}(g\mu + \sum_{i=1}^{f} \log(\nu_i))$.

Corollary [B.–Page]

Let F be a totally real field of odd degree, $\eta \subset \mathbb{Z}_F$ a level, $k \geq 2$ a weight and g the genus of $\Gamma_\eta \backslash \mathfrak{h}$. There exists an algorithm that, given a side pairing (σ_1, φ^*) and a face permutation σ_2 associated with a convex fundamental polygon for Γ_η, computes a basis of $H^1(\Gamma_\eta, W_k(\mathbb{C}))$ as a straight line program in time $\tilde{O}(gk^3)$.

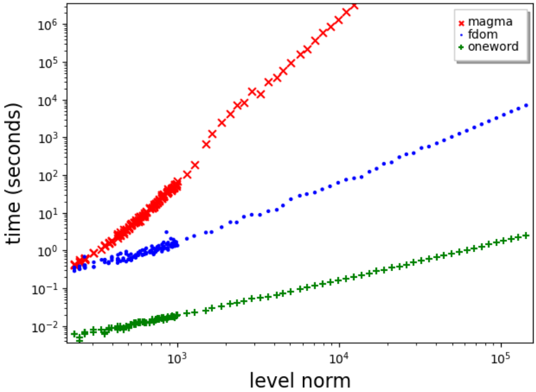
Timing comparison

Implementation in PARI/GP.

- red $\times$: Magma implementation of the indefinite method for Hilbert modular forms following Greenberg–Voight [GV11] with slope 4.18.
- blue $\bullet$: PARI implementation of fundamental domains for arithmetic Fuchsian groups following Rickards [Ric22] with slope 1.54.
- green $+$: our GP implementation of presentations for co-compact Fuchsian groups with slope 1.00.

As of now we have implemented one word and 1-handle presentations for co-compact Fuchsian groups.

Timing comparison



Work in progress

Hecke operators

From Greenberg–Voight [GV11], straight line programs are suitable for Hecke operators. But not implemented yet. May not be sparse in special basis.

Coverings

Avoiding precomputation of fundamental domain when increasing level. Expected in quasi-linear time. Mostly implemented.

Summary (Thanks for listening !)

- Can compute $H^1(\Gamma, W_k(\mathbb{C}^2))$ in time $\tilde{O}(\mu k^3)$.
- Can compute one word and one handle presentations of Γ in time $\tilde{O}(\mu)$.
- For $k = 2$, one word presentation is sufficient. For $k > 2$, need one handle presentation.
- For $\tilde{O}(\mu)$ space complexity, need straight line progams.
- Representation of basis of H^1 suitable for Hecke operators.

Special presentations for Γ using algebraic topology.

Let Γ be a co-compact Fuchsian group with signature $(g, \nu_1, \ldots, \nu_f)$ and Y the points of $\mathfrak{h}$ with non-trivial stabilizers by Γ. Then Γ acts totally discontinuously on $\mathfrak{h} - Y$. Fixing a vertex $v \in \mathfrak{h}$, by covering space theory $\Gamma = \Gamma(p)$ is the automorphism group of the covering

$$p \colon \mathfrak{h} - Y \to \Gamma \backslash (\mathfrak{h} - Y)$$

and there is a short exact sequence

$$1 \to \pi_1(\mathfrak{h} - Y, v) \to \pi_1(\Gamma \backslash (\mathfrak{h} - Y), \Gamma.v) \to \Gamma(p) \to 1$$

where a loop γ at $\Gamma.v$ lifting to the only path $\tilde{\gamma}$ with $\tilde{\gamma}(1) = v$ maps to the only $g \in \Gamma$ with $g.v = \tilde{\gamma}(0)$.

We only need to compute a good presentation for $\pi_1(\Gamma \backslash (\mathfrak{h} - Y), \Gamma.v)$ and push it to Γ.

Special presentations for Γ using algebraic topology.

Special presentations of $\pi_1(\Gamma\backslash(\mathfrak{h} - Y), \Gamma.v)$

Recall $X_\Gamma = \Gamma\backslash\mathfrak{h}$. From the injection $\Gamma\backslash(\mathfrak{h} - Y) \hookrightarrow X_\Gamma$ we get a surjection

$$\pi_1(\Gamma\backslash(\mathfrak{h} - Y), \Gamma.v) \to \pi_1(X_\Gamma, \Gamma.v)$$

but X_Γ is a compact orientable topological surface for which many combinatorial tools exist.

Idea

Compute a special presentation for $\pi_1(X_\Gamma, \Gamma.v)$ and lift it to $\pi_1(\Gamma\backslash(\mathfrak{h} - Y), \Gamma.v)$.

In [Bra21] Brahana gives tools and a procedure to obtain an explicit geometric presentation of $\pi_1(X_\Gamma)$ given any combinatorial embedding, but this procedure is not algorithmic. In [Imb01; Imb99] Imbert gives another procedure better suited for computations and extends it to geometric presentations of finitely generated Fuchsian groups from a fundamental domain. We make it into an actual algorithm and give a rigorous complexity analysis.

Special presentations for Γ using algebraic topology.

Recall $X_\Gamma = \Gamma\backslash\mathfrak{h}$. From the injection $\Gamma\backslash(\mathfrak{h} - Y) \hookrightarrow X_\Gamma$ we get a surjection

$$\pi_1(\Gamma\backslash(\mathfrak{h} - Y), \Gamma.v) \to \pi_1(X_\Gamma, \Gamma.v)$$

but X_Γ is a compact orientable topological surface for which many combinatorial tools exist.

Idea

Compute a special presentation for $\pi_1(X_\Gamma, \Gamma.v)$ and lift it to $\pi_1(\Gamma\backslash(\mathfrak{h} - Y), \Gamma.v)$.

In [Bra21] Brahana gives tools and a procedure to obtain an explicit geometric presentation of $\pi_1(X_\Gamma)$ given any combinatorial embedding, but this procedure is not algorithmic. In [Imb01; Imb99] Imbert gives another procedure better suited for computations and extends it to geometric presentations of finitely generated Fuchsian groups from a fundamental domain. We make it into an actual algorithm and give a rigorous complexity analysis.

Special presentations for Γ using algebraic topology.

Combinatorial tools

In [Tib26], Radó Tibor proves that every orientable compact surface X admits a combinatorial triangulation. Equivalently, admits a **combinatorial graph embedding** $\iota\colon G \hookrightarrow X$, i.e. $X - G$ is a disjoint union of polygons with edges E such that:

- Any edge $e \in E$ is glued to a unique other edge $\sigma_1(e) \neq e$.

Side pairing

The fixed-point free involution σ_1 is the **side pairing** of ι. A cycle of σ_1 is an **edge** of ι.

The orientation of X yields orientations on each polygons of $X - G$, equivalently cyclic orderings on the set of edges of each polygon.

Face permutation

The cyclic orderings correspond to a **face** permutation σ_2 of E. A cycle of σ_2 is a **face** of ι.

Face representation

The tuple (σ_1, σ_2) is a **face representation** of the embedding ι.

Instead of writing a full face representation

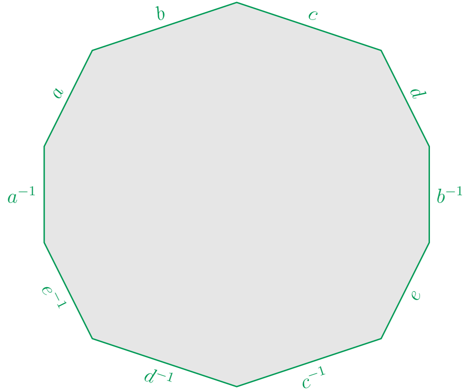
$$\sigma_1 = (a\bar{a})(b\bar{b})(c\bar{c})(d\bar{d})(e\bar{e})$$
$$\sigma_2 = (abcd\bar{b}e\bar{c}\bar{d}\bar{e}\bar{a})$$

we may just write the tuple of face words $((abcd\bar{b}e\bar{c}\bar{d}\bar{e}\bar{a}))$ understanding that any edge e is paired with $\bar{e}$.

Special presentations for Γ using algebraic topology.

An example

Let P be the polygon represented by the cycle $(abcd\bar{b}e\bar{c}\bar{d}\bar{e}\bar{a})$:



and X the quotient surface. The graph embedding $G \hookrightarrow X$ obtained from $\partial P \to X$ has 4 vertices, 5 edges and 1 face so that X has genus 1.

Special presentations for Γ using algebraic topology.

Combinatorial tools

Assume given a combinatorial embedding $\iota\colon G \hookrightarrow X$ with face representation (σ_1, σ_2) on a set of edges E. Assume further that G has a single vertex v and let C be a set of words in the edges E with one word corresponding to each cycle of σ_2. Then by a Van-Kampen argument

$$\pi_1(X_\Gamma, v) = \langle e \in E \mid c \in C;\ e\sigma_1(e) \rangle.$$

Remark

We can't directly apply it to the last example as G had 4 vertices.

Special presentations for Γ using algebraic topology.

Combinatorial tools

If we are given a combinatorial embedding with one face $\iota \colon G \hookrightarrow X$, we can obtain a combinatorial embedding with one vertex as follows.

Vertex permutation

Let (σ_1, σ_2) be a face representation of ι, then $\sigma_0 = \sigma_2^{-1}\sigma_1$ is the vertex permutation of ι. Each cycle of σ_0 is a vertex of ι.

Dual graph embedding

There is a graph embedding $\iota^* \colon G^* \hookrightarrow X$ with face representation $(\sigma_1, \sigma_1\sigma_0\sigma_1) = (\sigma_1, \sigma_1\sigma_2^{-1})$.

Building G^*

Let $V(G^*)$ be the set of centers of the faces of ι. Each directed edge $e \in D(G)$ lies at the boundary of a face P with $\sigma_1(e) = \bar{e}$ in another face P' of ι. Let e^* be a path from the center of P to the center of P' crossing only e and $\bar{e}$ and put $E(G^*) = \{e^* \mid e \in E(G)\}$. Then $G^* \hookrightarrow X$ is the required embedding.

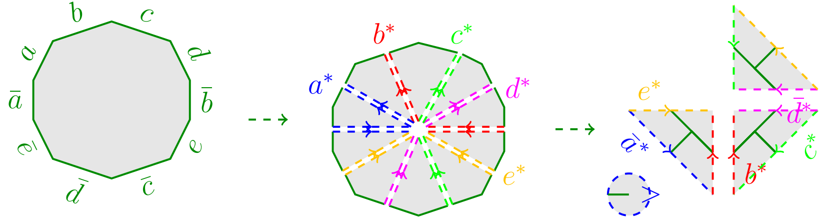
Special presentations for Γ using algebraic topology.

An example (II)

As in the last example let $\iota\colon G \hookrightarrow X$ have face representation

$$(abcd\bar{b}e\bar{c}\bar{d}\bar{e}\bar{a})$$

then the dual graph embedding $\iota^*\colon G^* \hookrightarrow X$is obtained as follows:



and has face representation

$$((a^*),\ (\bar{a^*}e^*\bar{b^*}),\ (b^*\bar{d^*}c^*),\ (d^*\bar{c^*}\bar{e^*}))$$

Special presentations for Γ using algebraic topology.

An example (III)

As ι^* has one vertex v, we obtain:

$$\pi_1(X, v) = \langle a, b, c, d, e \mid a,\ a^{-1}eb^{-1},\ bd^{-1}c,\ dc^{-1}e^{-1} \rangle.$$

Special presentations for Γ using algebraic topology.

Fundamental groups of surfaces: one face reduction.

Let $G^* \hookrightarrow X$ have one vertex and f faces $(P_i)_{i=1,\ldots,f}$.

One face reduction

Let P be the polygon obtained by gluing the faces along a spanning tree T in the graph with vertices $\{P_i\}_i$ and an edge e between P_i and P_j if $e \subset P_i \cap P_j$. Then $\partial P \to X$ induces a combinatorial embedding $G' \hookrightarrow X$ with one face and one vertex: the one face reduction of $G^* \hookrightarrow X$ along T.

In this case the induced presentation of $\pi_1(X, V(G'))$ is of the form

$$\pi_1(X, V(G')) \simeq \langle e_1, \ldots, e_n \mid \prod_{i=1}^{2n} e_{\varphi(i)}^{\epsilon_i} \rangle.$$

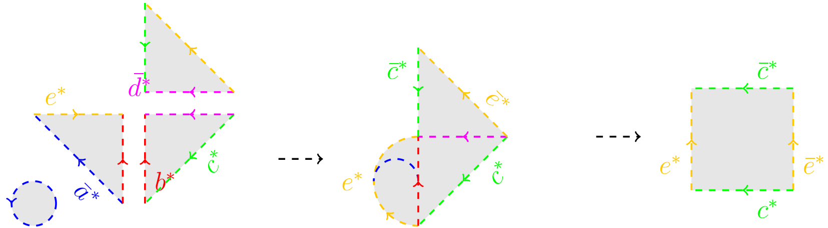
Special presentations for Γ using algebraic topology.

An example (IV): one face reduction.

The one face reduction ι_T of

$$((a^*),\ (\bar{a}^* e^* \bar{b}^*),\ (b^* \bar{d}^* c^*),\ (d^* \bar{c}^* \bar{e}^*))$$

along T with $E(T) = \{a^*, b^*, d^*\}$:



The embedding ι_T has face representation

$$(c^* e^* \bar{c}^* \bar{e}^*)$$

in particular, it is a torus.

Special presentations for Γ using algebraic topology.

Fundamental groups of surfaces: cut-and-paste.

Let w be a word representing a combinatorial embedding ι with one face and one vertex.

Cut-and-paste.

Assume $w = a\alpha b\beta\bar{a}\gamma\bar{b}\delta$. By cutting w along a and pasting along b we mean building a new embedding with face representation $w' = ac\bar{a}\gamma\beta\bar{c}\alpha\delta$. Then $c = \alpha b\beta$ and $b = \bar{\alpha}b\bar{\beta}$.

When w has one face and one vertex, such a and b always exist.

Theorem: [Bra21]

Let g be the genus of X. By a sequence of cut and pastes, we can obtain a word representing X of the form $w' = \prod_{i=1}^{m}[a_i, b_i]w''$ for any $0 \leq m \leq g$.

Special presentations for Γ using algebraic topology.

Fundamental groups of surfaces: cut-and-paste.

Let w be a word representing a combinatorial embedding ι with one face and one vertex.

Cut-and-paste.

Assume $w = a\alpha b\beta\bar{a}\gamma\bar{b}\delta$. By cutting w along a and pasting along b we mean building a new embedding with face representation $w' = ac\bar{a}\gamma\beta\bar{c}\alpha\delta$. Then $c = \alpha b\beta$ and $b = \bar{\alpha}b\bar{\beta}$.

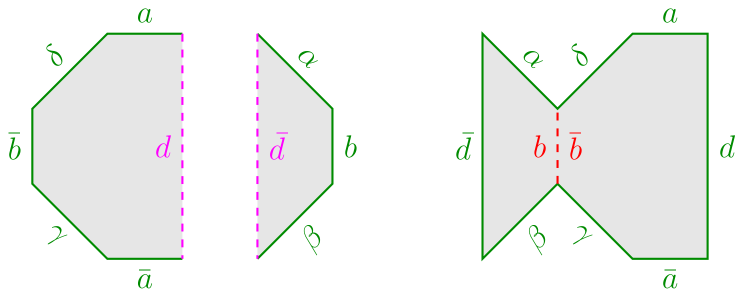
When w has one face and one vertex, such a and b always exist.

Theorem: [Bra21]

Let g be the genus of X. By a sequence of cut and pastes, we can obtain a word representing X of the form $w' = \prod_{i=1}^{m}[a_i, b_i]w''$ for any $0 \leq m \leq g$.

Special presentations for Γ using algebraic topology.

Cutting $a\alpha b\beta\bar{a}\gamma\bar{b}\delta$ along a and pasting along b



to obtain $ad\bar{a}\gamma\beta\bar{d}\alpha\delta$.

Back to Γ

Special presentations for Γ using algebraic topology.

Summary of the framework

Let P be a fundamental polygon for Γ with side pairing (σ_1, φ^*) and face permutation σ_2. We wish to compute $\pi_1(\Gamma\backslash(\mathfrak{h} - Y))$.

- We can't use the embedding $G \hookrightarrow X_\Gamma$ as $\Gamma\backslash Y \subset V(G)$.
- If $G^* \hookrightarrow X_\Gamma$ is the dual embedding, by construction, it has one vertex and points of $\Gamma\backslash Y$ all lie in distinct faces of ι^*. In particular there is a surjection

  $$\pi_1(G^*, V(G^*)) \to \pi_1(\Gamma\backslash(\mathfrak{h} - Y), V(G^*))$$

  with kernel those simple loops around points of $V(G) - \Gamma\backslash Y$.

We lift special presentations of $\pi_1(X_\Gamma, V(G^*))$ to special presentations of $\pi_1(G^*, V(G^*))$.

Special presentations for Γ from algebraic topology.

Lifting a special presentation

Assuming again G^* has one vertex. As

$$\pi_1(G^*, V(G^*)) = \pi_1(X_\Gamma - V(G), V(G^*)),$$

if $V(G) = \{c_i\}_i$ is the set of cycles of σ_2^*, δ_i is the simple loop $\bar{c}_i$ and $\langle E(G^*)|V(G)\rangle$ is the induced presentation on X_Γ, then by Van-Kampen

$$\pi_1(G^*, V(G^*)) = \langle e^* \in E(G^*), \delta_i; i \mid c_i\delta_i; i\rangle$$

and $\ker(\pi_1(G^*, V(G^*)) \to \pi_1(X_\Gamma, V(G^*))) = \langle \gamma.\gamma_i.\bar{\gamma}; i,\ \gamma \in \pi_1(G^*, V(G^*))\rangle$.

Special presentations for Γ from algebraic topology.

Lifting a special presentation (II)

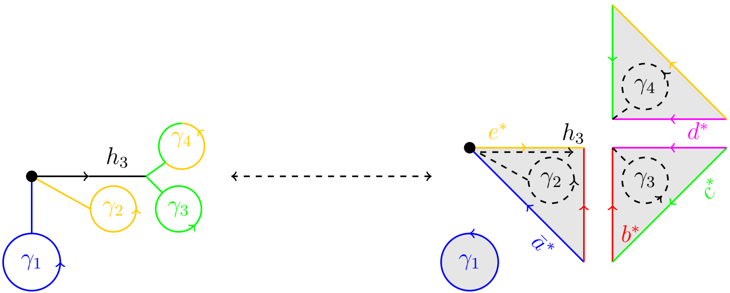
As cutting and pasting in X_Γ doesn't change the word representing X_Γ, it suffices to lift the one-word presentation. Let $G_T \hookrightarrow X_\Gamma$ be the gluing of G^* along a spanning tree T, represented by a word w. Let $\gamma = \bar{w}$, then there are some paths g_i such that if $\gamma_i = g_i \delta_i \bar{g}_i$ then $\gamma = \prod_{i=1}^{f} \gamma_i$ and we have

$$\pi_1(G^*, V(G^*)) = \langle e^*,\ \gamma_i ; e^* \in E(G_T) \mid w \prod_{i=1}^{f} \gamma_i \rangle$$

Special presentations for Γ from algebraic topology.

An example (VI): writing $\gamma = \prod_i \gamma_i$.

Let $\gamma_1 = \bar{a}$, $\gamma_2 = a\bar{b}\bar{e}$, $h_3 = e^*$, $\gamma_3 = h_3.(b\bar{c}d).\overline{h_3}$ $\gamma_4 = h_3.(\bar{d}ec).\bar{h}_3$.



After gluing along $T = a^* \cup b^* \cup d^*$ we find that

$$\gamma = \gamma_1.\gamma_2\gamma_3\gamma_4 = \bar{c}.ec\bar{e}.$$

Special presentations for Γ from algebraic topology.

An example (VII).

Let

- $F = \mathbb{Q}(\sqrt{8})$,
- v the non-trivial place of F,
- B the quaternion algebra $\left(\frac{-\sqrt{8}/2,-3}{F}\right)$,
- O a maximal order in B.

Then B embeds in $B \otimes_F F_v \simeq M_2(\mathbb{R})$ yielding an embedding $O^1 \hookrightarrow \mathrm{SL}_2(\mathbb{R})$ of norm 1 units in O. Let Γ be the projective image of O^1 in $\mathrm{PSL}_2(\mathbb{R})$. Using Rickards software [Ric22] we find a fundamental domain for Γ represented by

$$(abcd\bar{b}e\bar{c}\bar{d}\bar{e}\bar{a})$$

yielding the previous embedding $G \hookrightarrow \Gamma\backslash\mathfrak{h}$. Using the dual embedding, we find

$$\pi_1(\Gamma\backslash(\mathfrak{h}-Y), V(G^*)) = \langle a^*, b^*, c^*, d^*, e^* \mid a^*\gamma_1,\ \overline{a^*}e^*\overline{b^*}\gamma_2,\ b^*\overline{d^*}c^*\gamma_3,\ d^*\overline{c^*}\overline{e^*}\gamma_4\rangle$$

Special presentations for Γ from algebraic topology.

An example (VIII).

Lifting along the tree $T = a^* \cup b^* \cup d^*$ we find that

$$\Gamma = \langle c, d, \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid [c, d]\gamma_1\gamma_2\gamma_3\gamma_4 = 1, \gamma_1^3 = 1\rangle$$

Work in progress

For a covering $\Gamma_1 \backslash \mathfrak{h} \to \Gamma_2 \backslash \mathfrak{h}$ given as a permutation representation $\Gamma_1 \to S_n$. Compute one-word, m-handles and geometric presentations for Γ_2 from that of Γ_1 expected in $\tilde{O}(n)$ time. Permits increasing the level of $S_k(\eta)$ in quasi-linear time in the norm of the level without computing new fundamental domains.

Thanks!

[Bra21] H. R. Brahana. "Systems of circuits on two-dimensional manifolds". In: *Ann. of Math. (2)* 23.2 (1921), pp. 144–168. ISSN: 0003-486X,1939-8980. DOI: 10.2307/1968030. URL: https://doi.org/10.2307/1968030.

[Tib26] Radó Tibor. "Über den Begriff der Riemannschen Fläche". In: *Acta litterarum ac scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae : Sectio scientiarum mathematicarum* (1926).

[Man73] Yu. I. Manin. "Parabolic points and zeta-functions of modular curves". English. In: *Math. USSR, Izv.* 6 (1973), pp. 19–64. ISSN: 0025-5726. DOI: 10.1070/IM1972v006n01ABEH001867.

[Imb99] Michel Imbert. "Fundamental groups of fat-graphs". In: *Note Mat.* 19.2 (1999), pp. 257–268. ISSN: 1123-2536,1590-0932.

[Imb01] Michel Imbert. "Calculs de présentations de groupes fuchsiens via les graphes rubanés". In: *Expo. Math.* 19.3 (2001), pp. 213–227. ISSN: 0723-0869. DOI: 10.1016/S0723-0869(01)80002-5. URL: https://doi.org/10.1016/S0723-0869(01)80002-5.

[GV11] Matthew Greenberg and John Voight. "Computing systems of Hecke eigenvalues associated to Hilbert modular forms". In: *Math. Comp.* 80.274 (2011), pp. 1071–1092. ISSN: 0025-5718,1088-6842. DOI:

10.1090/S0025-5718-2010-02423-8. URL:
https://doi.org/10.1090/S0025-5718-2010-02423-8.

[Ric22] James Rickards. "Improved computation of fundamental domains for arithmetic Fuchsian groups". In: *Math. Comp.* 91.338 (2022), pp. 2929–2954. ISSN: 0025-5718,1088-6842. DOI: 10.1090/mcom/3777. URL: https://doi.org/10.1090/mcom/3777.