# On the isomorphism problem of nf-extensions

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COHOMOLOGY GEOMETRY AND EXPLICIT NUMBER THEORY

## The problem

Let $K$ **be a number field** with maximal order $\mathbb{Z}_K$ and we fix a **subgroup $U$ of the units $\mathbb{Z}_K^\times$** generating the field, i.e. $\mathbb{Q}U = K$.
Now let **$I$ be a fractional $\mathbb{Z}U$-ideal**. With this we can construct group extensions

$$\mathbf{0} \longrightarrow \boldsymbol{I} \longrightarrow \boldsymbol{E} \longrightarrow \boldsymbol{U} \longrightarrow \mathbf{1}$$

where the conjugation action of $U$ on $I \trianglelefteq E$ is given via the **natural multiplication of $U$ on $I \leq K$**. In other words, we have $E = E(\tau)$ for some cocycle $\tau \in Z^2(U, I)$. We call the constructed group $E$ an **nf-extension of $U$ by $I$**.

Our goal is to classify all such extensions up to isomorphism.

Note that these kinds of groups naturally arise when looking at group extensions of the form $\mathbb{Z}^n \hookrightarrow E \twoheadrightarrow A$ where $A$ is a f.g. abelian group acting irreducibly and faithfully on $\mathbb{Z}^n$. Namely, the resulting representation $\rho : A \hookrightarrow \mathrm{GL}(n, \mathbb{Z})$ gives $\mathbb{Q}^n$ a vector-space structure over the field $K := \mathbb{Q}(\rho(A))$ with $\rho(A) \leq \mathbb{Z}_K^\times$.

We denote the set of isomorphism classes by

$$\mathcal{E}(U, I) := \{E \mid E \text{ an nf-extension of } U \text{ by } I\}/\cong .$$

Immediately, we have a natural surjection

$$H^2(U, I) \twoheadrightarrow \mathcal{E}(U, I).$$

Fixing only $U$, we denote the set of isomorphism classes by

$$\mathcal{E}(U) := \{E \mid E \text{ an nf-extension of U by any fractional ideal } I\}/\cong .$$

Taking $I_1, \dots, I_l$ to be representative ideals of the ideal class monoid $ICM(\mathbb{Z}U)$, we have a natural surjective map

$$\bigsqcup_{i=1}^{l} \mathcal{E}(U, I_i) \twoheadrightarrow \mathcal{E}(U).$$

## Isomorphisms between nf-extensions and compatible pairs

Any group isomorphism $\varphi : E \to E'$ between nf-extensions $E \in \mathcal{E}(U, I)$ and $E' \in \mathcal{E}(U', I')$ induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "\mu"] & E \arrow[r] \arrow[d, "\varphi"] & U \arrow[r] \arrow[d, "\nu"] & 1 \\
0 \arrow[r] & I' \arrow[r] & E' \arrow[r] & U' \arrow[r] & 1
\end{tikzcd}$$

where $\mu$ and $\nu$ are isomorphisms satisfying the equation

$$\mu(a \cdot u) = \mu(a) \cdot \nu(u), \qquad \text{for all } a \in I,\, u \in U.$$

We call such a pair $(\nu, \mu)$ of isomorphisms a **compatible pair** and denote the set of compatible pairs by $\mathrm{Comp}((U, I), (U', I'))$. Taking $\tau \in Z^2(U, I)$ and $\tau' \in Z^2(U', I')$ with $E = E(\tau)$ and $E' = E(\tau')$ this implies

$$\tau^{(\nu,\mu)}(u', v') := \mu\left(\tau\left(\nu^{-1}(u'), \nu^{-1}(v')\right)\right) = \tau'(u', v'), \qquad \text{for all } u', v' \in U'.$$

It turns out that **two nf-extensions $E = E(\tau)$ and $E' = E(\tau')$ are isomorphic to each other if and only if there exists a compatible pair $(\nu, \mu)$ such that $\tau^{(\nu,\mu)} = \tau'$.**

The set of compatible pairs $\mathrm{Comp}((U, I), (U', I'))$ is given by group isomorphisms $(\nu : U \to U', \mu : I \to I')$ where **$\nu : K = \mathbb{Q}U \to \mathbb{Q}U' = K$ defines a field isomorphism** and $\mu$ is of the form

$$\mu(a) = c_\mu \cdot \nu(a), \qquad \text{for all } a \in I,$$

with $c_\mu \in K^\times$ such that $c_\mu \cdot \nu(I) = I'$.

## Algorithm to compute $\mathcal{E}(U, I)$

1. Calculate the second cohomology group $H^2(U, I)$.
2. Calculate the group of compatible pairs $\mathrm{Comp}(U, I)$.
3. Compute the orbits of the $\mathrm{Comp}(U, I)$-action on $H^2(U, I)$ and take one representative $\tau_i \in H^2(U, I)$ from each orbit.

$\Rightarrow$ We have $\mathcal{E}(U, I) = \{E(\tau_1), \dots, E(\tau_n)\}$.

## Algorithm to compute $\mathcal{E}(U)$

1. Calculate $\mathrm{Fix}(U) := \{\nu \in \mathrm{Aut}(K) \mid \nu(U) = U\}$.
2. Obtain representative ideals $I_1, \dots, I_l$ of the ideal class monoid $\mathrm{ICM}(\mathbb{Z}U)$.
3. Compute the orbits of $\mathrm{Fix}(U)$ acting on $\mathrm{ICM}(\mathbb{Z}U)$ and take one representative $J_j$ from each orbit.
4. Calculate $\mathcal{E}(U, J_j)$ for $j = 1, \dots, m$.

$\Rightarrow$ We have $\mathcal{E}(U) = \bigsqcup_j \mathcal{E}(U, J_j)$.

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MSCA-DN Grant #101169527

Funded by the European Union

UK Research and Innovation