On Integers Representable as the Sum of Two Units

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Introduction

Let K be a number field with maximal order $\mathcal{O}_K$. We want to classify pairs of units $\varepsilon, \delta \in \mathcal{O}_K^\times$ and integers $n \in \mathbb{Z}$ satisfying

$$\varepsilon + \delta = n.$$

Siegel [4] proved that, for any fixed number field K and fixed nonzero integer $n \in \mathbb{Z}$, there are only finitely many units $\varepsilon, \delta \in \mathcal{O}_K^\times$ such that $\varepsilon + \delta = n$, with Baker's [2] results on linear forms in logarithms providing effective algorithms to compute all such units (ε, δ) over any number field K.

Let $\mathcal{N}_K$ denote the set of all positive integers $n \in \mathbb{Z}$ such that there exist units $\varepsilon, \delta \in \mathcal{O}_K^\times$ satisfying $\varepsilon + \delta = n$. We note that $\mathcal{N}_K$ can be infinite if K contains a real quadratic subfield L, e.g. if ε_L denotes the fundamental unit of L, then for all integers $i \geq 1$, we have $\mathrm{Tr}_{L/\mathbb{Q}}(\varepsilon_L^i) \in \mathcal{N}_K$, due to the unit solution

$$\varepsilon_L^i + \overline{\varepsilon}_L^i = \mathrm{Tr}_{L/\mathbb{Q}}(\varepsilon_L^i).$$

Otherwise, if K does not contain any real quadratic subfield, then $\mathcal{N}_K$ is finite, as proven by Tinková, Yatsyna, and the first author [5]. However, the proof in [5] relies on the finiteness of solutions to unit equations in several unknowns and thus cannot effectively determine the set $\mathcal{N}_K$ for an arbitrary number field K.

Our work aims to give a fully effective algorithm to compute $\mathcal{N}_K$ for a given suitable number field K, building on recent work of Vukusic and the second author [6] and Komatsu [3] in the case of cyclic cubic fields K.

Main results

Theorem 1

Let K be a number field. Suppose that for any nonzero $n \in \mathbb{Z}$, there is no unit $\varepsilon \in \mathcal{O}_K^\times$ such that $n - \varepsilon$ is a Galois conjugate of ε. If $n \neq 0$ is a rational integer and $\varepsilon, \delta \in \mathcal{O}_K^\times$ are units such that $n = \varepsilon + \delta$, then

$$\max\{\log|n|, h(\varepsilon), h(\delta)\} \leq C$$

where C is an **effectively computable** constant depending only on K. Here h is the absolute logarithmic Weil height.

Corollary 1

Assume K is a number field such that, for every subfield $L \subseteq K$ either its absolute degree $d = [L : \mathbb{Q}]$ is odd, or the Galois group of its normal closure, viewed as a transitive permutation group of degree d, is not contained in the wreath product $C_2 \wr S_{d/2}$. Then $\mathcal{N}_K$ is an effectively comptuble finite set.

In particular, $\mathcal{N}_K$ is effectively comptutable if either $D = [K : \mathbb{Q}]$ is odd, or $D \geq 3$ and K is primitive.

Bounds for the constant C

To give an explicit example of a constant C, we can prove the bound

$$C := \max_{L \subseteq K} \left\{2A_L^2 B_L \max\left\{e \cdot r_L t_L (A_L B_L + 1),\ 4B_L \log(4r_L) + 4C_L \log(2C_L)\right\}\right\},$$

where the maximum is taken over all subfields L of K, and where for each subfield $L \subseteq K$, we define the real constants A_L, B_L, C_L as:

$$A_L := t_L \cdot \max_{i=1,\dots,t_L} \left\{\log\left|\eta_i^{(1)}\right|\right\}, \quad B_L := \frac{t_L}{\mathrm{Reg}_L} \cdot \max_{1 \leq i,j \leq t_L} \{|\det R_{L,i,j}|\}, \quad \text{and}$$

$$C_L := B_L \cdot 2^{12t_L+27} D^2 (D-1)^2 \big(1 + \log(D(D-1))\big) \cdot \pi \left(\prod_{k=1}^{t_L} h_m(\eta_k)\right)^2,$$

where D is the degree of K, t_L is the unit rank of $\mathcal{O}_L^\times$, r_L is the order of the torsion subgroup of $\mathcal{O}_L^\times$, $\eta_1, \eta_2, \dots, \eta_{t_L} \in \mathcal{O}_L^\times$ are a fundamental system of units for L, Reg_L is the regulator of L, $R_{L,i,j}$ is the matrix formed by removing the i-th column and j-th row from the matrix $R_L := (\log|\eta_i^{(j)}|)_{1 \leq i,j \leq t_L}$, and $h_m(\cdot)$ is the modified height,

$$h_m(\alpha) := \max\{Dh(\alpha), |\log\alpha|, 0.16\}.$$

Sketch Proof

1. Assume that $\varepsilon + \delta = n$, and let $\varepsilon = \eta_1^{x_1} \cdots \eta_t^{x_t}$ and $\delta = \eta_1^{y_1} \cdots \eta_t^{y_t}$, where $\eta_1, \dots, \eta_t \in \mathcal{O}_L^\times$ is a fundamental system of units for $L = \mathbb{Q}(\varepsilon)$. Let
$$X := \max_{i=1,\dots,t}\{|x_i|\}, \quad \text{and} \quad Y := \max_{i=1,\dots,t}\{|y_i|\}.$$
2. Let $\varepsilon^{(1)}, \dots, \varepsilon^{(d)}$ and $\delta^{(1)}, \dots, \delta^{(d)}$ be the d distinct embeddings of ε and δ resp. into $\mathbb{C}$. For some fixed $i < j$, let m be the second smallest element of $|\varepsilon^{(i)}|$, $|\delta^{(i)}|$, $|\varepsilon^{(j)}|$, $|\delta^{(j)}|$. By using the equation
$$\varepsilon^{(i)} + \delta^{(i)} - \varepsilon^{(j)} - \delta^{(j)} = 0$$
and applying the unit matching techniques of Bajpai and Bennett [1], we prove there exist effectively computable constants C_1 and C_2 such that either $m \geq \frac{|\varepsilon^{(i)}|}{4r} \exp(-C_1 \log X)$ or $X \leq C_2$.
3. Using the lower bound on m, we show that $|\varepsilon^{(1)}| \leq C_3 + C_4 \log X$ for some effective constants C_3, C_4. This inequality gives an effective bound on X, and thus on $h(\varepsilon)$ (and similarly for $h(\delta)$).

Worked example: Degree 5 field

Let $K = \mathbb{Q}(\alpha)$ where α is a root of the degree 5 polynomial $x^5 - 5x^3 - x^2 + 5x + 1$. As $[K : \mathbb{Q}]$ is odd, we can effectively compute $\mathcal{N}_K$.

Theorem 2

There are exactly 255 non-equivalent pairs of units $\varepsilon, \delta \in \mathcal{O}_K^\times$ such that $\varepsilon + \delta$ is a rational non-zero integer n. In particular, we have

$$\mathcal{N}_K = \{1, 2, 3, 4, 5, 6, 7, 10, 11, 15, 251\}.$$

Computation sketch:

1. Compute the constant C from Theorem 1 to get:
$$\max\{h(\varepsilon), h(\delta)\} < 1.91 \cdot 10^{34}.$$
2. Apply seven rounds of LLL-reduction to get:
$$\max\{h(\varepsilon), h(\delta)\} < 111.49.$$
3. Enumerate over the remaining $1.68 \cdot 10^9$ solutions to find a total of 255 non-equivalent solutions (took ~ 23 CPU-days).

The largest solution found is

$$\varepsilon = +\eta_1^{-4}\eta_2^{10}\eta_3^{0}\eta_4^{-2}, \quad \delta = -\eta_1^{-1}\eta_2^{-8}\eta_3^{1}\eta_4^{6}, \quad \text{and} \quad n = 251,$$

where $\eta_1, \eta_2, \eta_3, \eta_4$ are a fundamental system of units for $\mathcal{O}_K^\times$.

Further problems

While we are still unfortunately unable to resolve the general number field case, we conjecture that effective height bounds exist for all solutions to $\varepsilon + \delta = n$, with the exception of the real quadratic solutions.

Conjecture

Let K be a number field, and let $\varepsilon, \delta \in \mathcal{O}_K^\times$ be units such that $\varepsilon + \delta = n$ for some nonzero $n \in \mathbb{Z}$, and assume that $\mathbb{Q}(\varepsilon)$ is not a real quadratic field. Then

$$\max\{\log|n|, h(\varepsilon), h(\delta)\} \leq C'$$

where C' is an **effectively computable** constant depending only on K.

References

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