

Existence of minimal del Pezzo surfaces of degree 1 with conic bundles over finite fields



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Introduction

Del Pezzo surface X over a field k : smooth projective geometrically integral surface such that $-K_X$ is ample.

Degree: $d := K_X^2$, satisfies $1 \leq d \leq 9$. We consider $d = 1$.

If $k = \bar{k}$: del Pezzo surface of degree 1 is a blow-up of $\mathbb{P}^2$ in 8 points in general position.

If $k = \mathbb{F}_q$, we classify del Pezzo surfaces X based on the conjugacy class of the Frobenius automorphism in $\text{Pic}(\bar{X})$. We call this the **type** of the surface.

For degree 1: 112 possible conjugacy classes [7], of which 37 are **minimal**, and 7 of those have a **conic bundle structure**.

Question

For which q does there exist a **minimal del Pezzo surface of degree 1 with conic bundle** of each of the 7 types over $\mathbb{F}_q$?

Classification

Standard conic bundle $f: X \rightarrow \mathbb{P}^1$ of degree 1 over $\bar{k}$: 7 singular fibers, which are pairs of (-1) -curves intersecting transversally in a point.

Over $\mathbb{F}_q$, the singular fibers lie above closed points, with the sum of their degrees equal to 7.

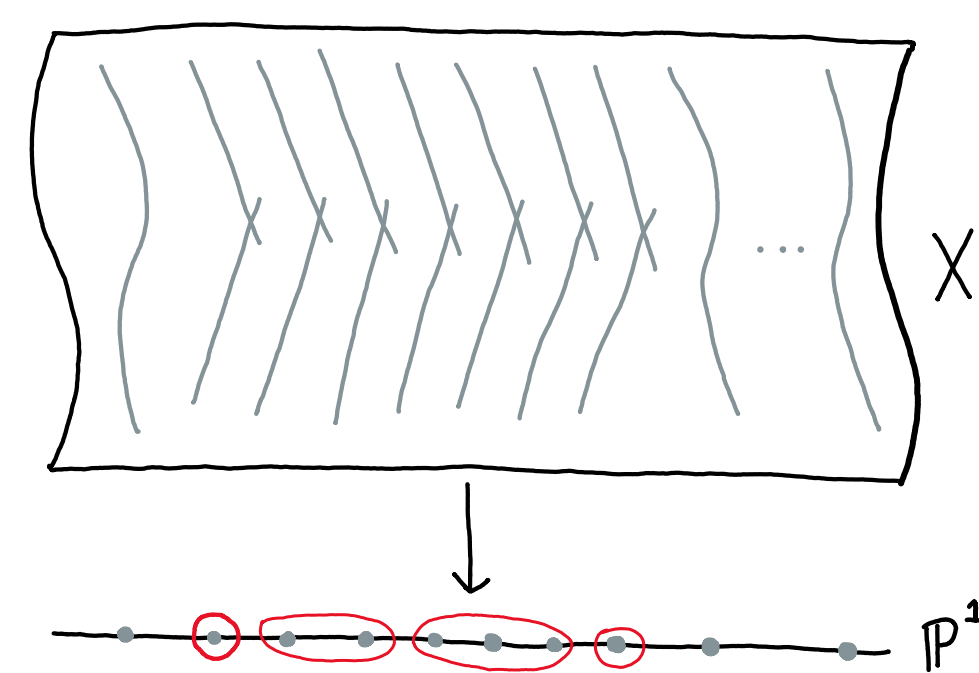


Figure 1. A standard rational conic bundle with 7 singular fibers.

Proposition 1 ([3]). *There exists a minimal standard conic bundle $f: X \rightarrow \mathbb{P}^1$ over $\mathbb{F}_q$ with singular fibers above closed points $x_1, \dots, x_s \in \mathbb{P}_{\mathbb{F}_q}^1$ if and only if s is even.*

Degree 1: $x_1, \dots, x_s \in \mathbb{P}_{\mathbb{F}_q}^1$ with s even, sum of degrees equal to 7.

There is a 1-1 correspondence:

Ways of partitioning 7 points into an even number of parts



7 types of minimal del Pezzo surfaces of degree 1 with conic bundle

- Type 1: five $\mathbb{F}_q$ -points, one point of degree 2;
- Type 2: two $\mathbb{F}_q$ -points, one point of degree 2, one point of degree 3;
- Type 3: one $\mathbb{F}_q$ -point, three points of degree 2;
- Type 4: three $\mathbb{F}_q$ -points, one point of degree 4;
- Type 5: one $\mathbb{F}_q$ -point, one point of degree 6;
- Type 6: one point of degree 2, one point of degree 5;
- Type 7: one point of degree 3, one point of degree 4.

Main result

There exists a del Pezzo surface of degree 1 over $\mathbb{F}_q$:

- of type 1 if and only if $q \geq 5$.
- of type 2 if and only if $q \geq 3$.
- of type 3 if $q \geq 4$, and it does not exist when $q = 2$.
- of type 4 if $q \geq 7$.
- of type 5 if $q \geq 4$.
- of type 6 for all q .
- of type 7 for all q .

Method 1

By Proposition 1, for the 7 types: If configuration of points exists in $\mathbb{P}_{\mathbb{F}_q}^1$, then conic bundle with singular fibers above those points exists.

Use existence of conic bundle to construct del Pezzo surface of same type, using method inspired by [5]. We use the following **elementary transformations** of $f: X \rightarrow \mathbb{P}^1$:

- Take $P \in X(\mathbb{F}_q)$ on smooth fiber F of f ;
- Blow up X in P ;
- Blow down strict transform of F .

The resulting surface is again a minimal standard conic bundle of the same type.

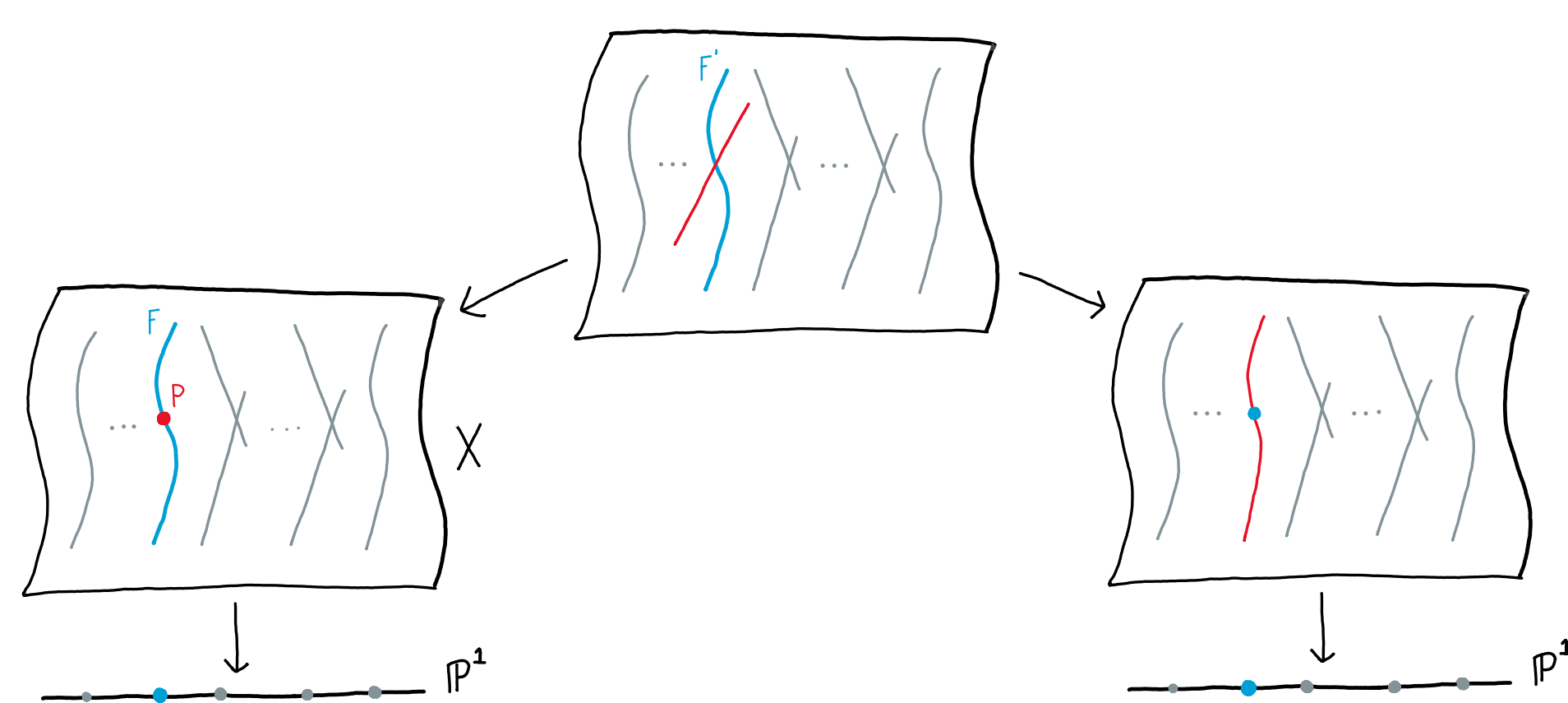
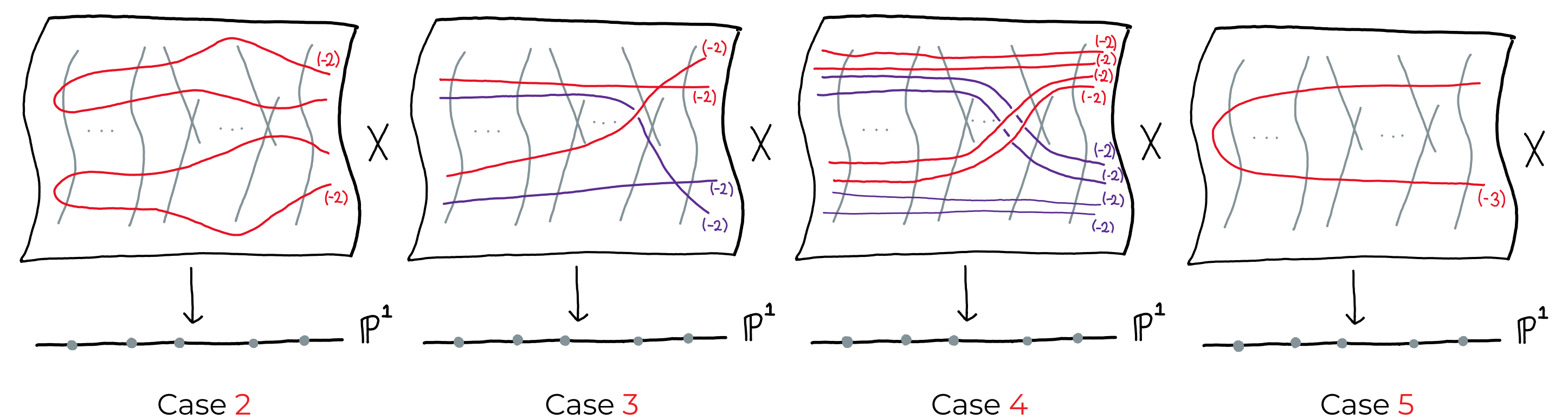


Figure 2. An elementary transformation: we blow up in P , and subsequently blow down F' .

Strategy

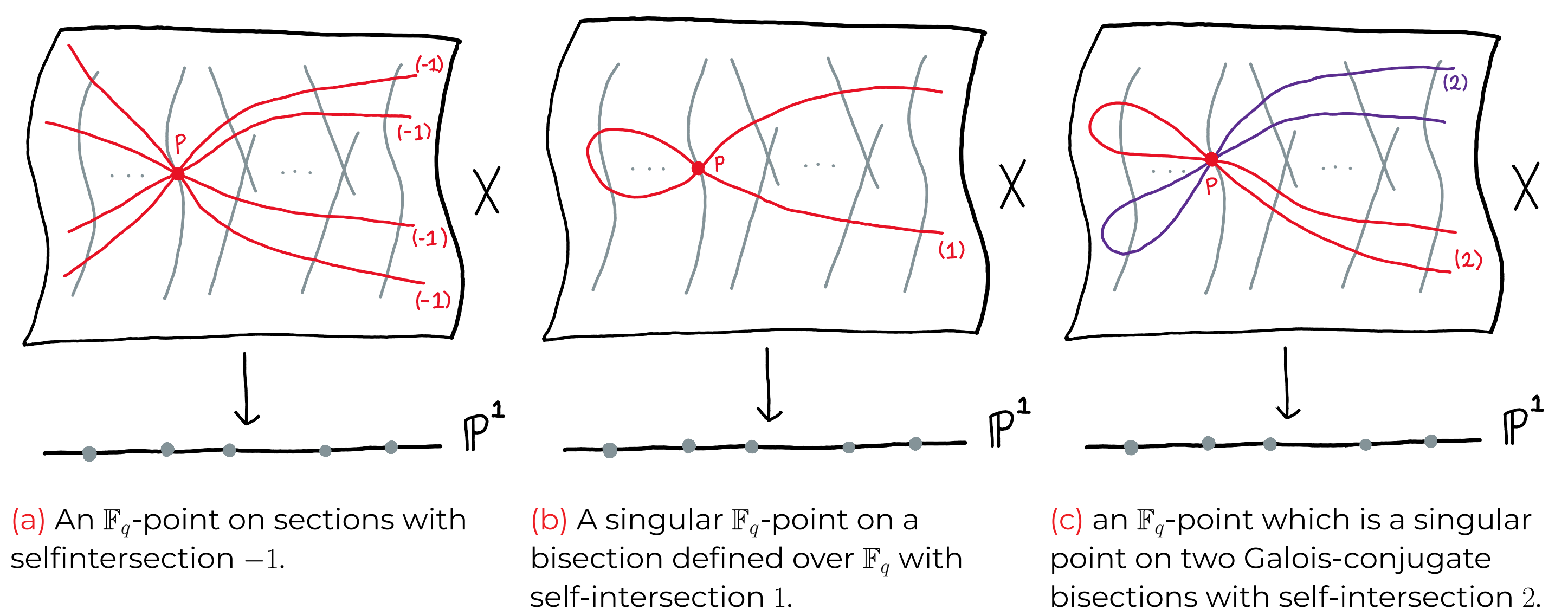
Proposition 2. *Let $f: X \rightarrow \mathbb{P}^1$ be a minimal standard conic bundle of degree 1 over a perfect field k . Then one of the following holds.*

1. The surface X is a **del Pezzo surface** of degree 1 with two conic bundle structures.
2. There are **two disjoint Galois-conjugate bisections** with self-intersection -2 .
3. There are **four Galois-conjugate sections**, each with self-intersection -2 .
4. There are **eight disjoint Galois-conjugate sections** with self-intersection -2 .
5. There is an irreducible **bisection** on X with self-intersection -3 .



We use elementary transformations as in Figure 2 to transform a conic bundle over $\mathbb{F}_q$ of case 2, 3, 4, 5 into a del Pezzo surface (case 1), if q is large enough.

Strategy: For each case, show that there exists an $\mathbb{F}_q$ -point P on a smooth fiber of f which avoids the following situations:



Perform elementary transformation at $P \rightsquigarrow$ no more curves of self-intersection < -1 .

For each case in Proposition 2, and each type, we give an explicit bound for the number of points to avoid per fiber, so if q is large enough, we can find an $\mathbb{F}_q$ -point P on a smooth fiber which avoids situations (a), (b) and (c).

Method 2

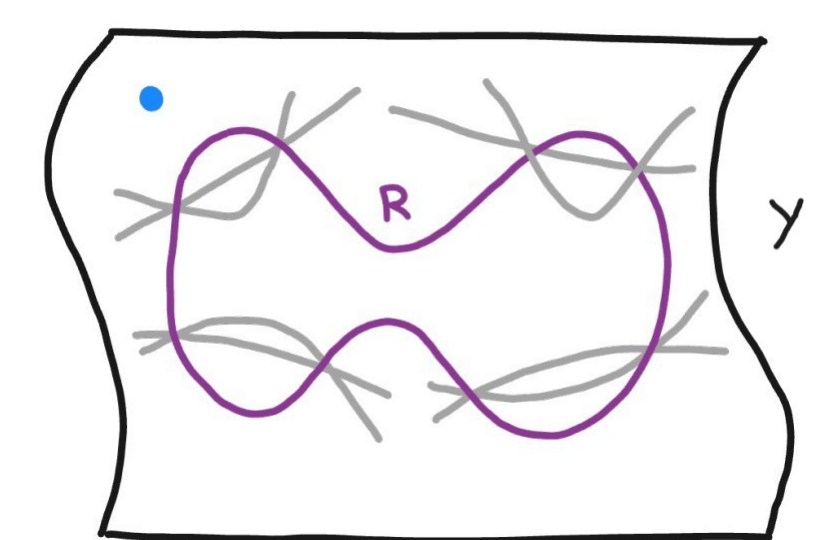
Each del Pezzo surface of degree 1 (dP1) over $\mathbb{F}_q$ has a unique quadratic twist called its **Bertini twist**, which is again a dP1 over $\mathbb{F}_q$.

Type 1, 2 and 4: Bertini twist has a (-1) -curve defined over $\mathbb{F}_q \rightsquigarrow$ can be blown down to del Pezzo surface of degree 2 (dP2).

We can derive the existence of such a dP1 from the existence of the corresponding dP2.

Existence of del Pezzo surfaces of degree 2 for each q is classified in [6].

- Let Y be a dP2 over $\mathbb{F}_q$.
- The morphism induced by $|-K_Y|$ has a ramification divisor R , which lies above a quartic curve in $\mathbb{P}^2$.
- Let $S := R \cup \{(-1)\text{-curves on } Y\}$.
- By [4]: $\text{Bl}_P Y$ is dP1 $\iff P \notin S$.
- $\#Y(\mathbb{F}_q) = q^2 + a(Y)q + 1$, and



$$\#S(\mathbb{F}_q) \leq \begin{cases} 2(q+1) + \#\{\mathbb{F}_q\text{-points on } (-1)\text{-curves on } Y\} & \text{if } q \text{ even;} \\ (q+1+6\sqrt{q}) + \#\{\mathbb{F}_q\text{-points on } (-1)\text{-curves on } Y\} & \text{if } q \text{ odd.} \end{cases}$$

- If $\#Y(\mathbb{F}_q) - \#S(\mathbb{F}_q) \geq 1 \rightsquigarrow \exists \mathbb{F}_q\text{-point outside } S \rightsquigarrow \text{dP1 exists.}$

For dP1 of index 8 (blow-up of $\mathbb{P}^2$): determine (non-)existence for small q computationally using Magma.

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