Sato-Tate groups of K3 Kummer Surfaces

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Introduction

The Sato–Tate conjecture was originally stated independently by Mikio Sato and John Tate for elliptic curves $E/\mathbb{Q}$ without complex multiplication. It predicts that the normalized traces of Frobenius are equidistributed in the interval $[-2, 2]$ with respect to the Sato–Tate measure

$$\frac{\sqrt{4-t^2}}{2\pi}\,dt.$$

This distribution can be interpreted as that of the trace of the Galois representation on the compact Lie group $\mathrm{SU}(2)$ associated with E. The group $\mathrm{SU}(2)$, equipped with this representation, is called the Sato–Tate group $\mathrm{ST}(E)$.

Numerous authors have generalized the study of Sato–Tate groups to the symplectic representations in $\mathrm{USp}(2g)$ associated to abelian varieties. The aim of this poster is to present the analogous orthogonal Sato-Tate groups associated to K3 surfaces, with a particular focus on Kummer surfaces.

Character Rings

The virtual character ring: Let G be a compact Lie group. The virtual character ring $\mathfrak{R}(G)$ is the free abelian group on irreducible complex-valued characters, equipped with an addition law given by the direct sum, and a multiplication law given by the tensor product. The virtual character ring of G is of the form $\mathfrak{R}(G) = \bigoplus_{n\geq 0}\mathbb{Z}\psi_n$ where ψ_n are the irreducible complex-valued characters of G.

Every character λ of G can be uniquely decomposed as a sum of irreducible characters, in particular there exist $n_i \in \mathbb{N}$ such that

$$\lambda = \sum_i n_i \psi_i,$$

hence can be identified with an element of $\mathfrak{R}(G)$.

Orthogonality relations: Let χ_1, χ_2 be irreducible characters of G. Then the orthogonality relations for characters:

$$\langle \chi_i, \chi_j \rangle = \begin{cases} 0 \text{ for } i \neq j, \\ 1 \text{ for } i = j. \end{cases}$$

extends bilinearly to $\mathfrak{R}(G) \times \mathfrak{R}(G) \to \mathbb{Z}$. We say that two characters χ_1 and χ_2 are independent when $\langle \chi_1, \chi_2 \rangle = 0$.

Sato-Tate groups of Surfaces

Abelian surfaces: Sato–Tate groups of abelian surfaces were studied by Fité, Kedlaya, Rotger, and Sutherland in their paper *Sato–Tate distributions and Galois endomorphism modules in genus 2*. They classify Sato–Tate groups of abelian surfaces as Lie subgroups of $\mathrm{USp}(4)$ and proved that there are 52 possibilities for the Sato–Tate groups of an abelian surfaces. These 52 compact Lie groups are partitioned into six classes of connected components.

K3 surfaces: A *K3 surface* over a field k is a smooth algebraic projective surface X such that

$$\omega_X \simeq \mathcal{O}_X$$

and

$$H^1(X, \mathcal{O}_X) = 0.$$

For a K3 surface, the Sato–Tate group $\mathrm{ST}(X)$ is a compact Lie subgroup of the orthogonal group

$$\mathrm{O}(22 - \rho(X)),$$

where

$$\rho(X) := \mathrm{rk}\big(\mathrm{Pic}(X)\big)$$

denotes the Picard rank of X.

General Framework for the Character Method

Given a K3 surface X of Picard rank $\rho(X)$, a family of characters of $\mathrm{O}(22-\rho(X))$, a sample of primes of good reduction, and a compact Lie group H proposed as a candidate for the Sato–Tate group of X, the strategy is to compute the normalized Frobenius characteristic polynomials and the associated character inner-product matrix. The resulting matrix is then compared with the one predicted by the character theory for H.

Kummer Surfaces

Let A an abelian surface. Denote the inverse map on A by $[-1] : A \longrightarrow A$. The fixed points of $[-1]$ are the sixteen points of the 2-torsion subgroup $A[2] \simeq (\mathbb{Z}/2\mathbb{Z})^4$, which give 16 singular points of the quotient surface $A/\langle[-1]\rangle$. Its desingularization is a K3 surface, called **the Kummer surface associated to** A, and denote by $\mathrm{Kum}(A)$.

The Kummer surface is obtained from $A/\langle[-1]\rangle$ by blowing up by the image of $A[2]$, which gives a smooth surface with sixteen exceptional lines.

Example: Let $X = \mathrm{Kum}(A)$ a Kummer surface associated by an abelian surface A. Then the geometric Picard rank of X is given by:

$$17 \leq \rho(X) \leq 20.$$

More specifically:

- If $A \sim E_1 \times E_2$, then $\rho(X) \geq 18$.
- If $A \sim E \times E$ where E is generic, then $\rho(X) \geq 19$.
- If $A \sim E \times E$ and E is a CM elliptic curve, then $\rho(X) = 20$.

Exceptional isogenies of Lie groups

The paradox that symplectic representations give rise to orthogonal representations is explained by the exceptional isogenies of Lie groups:

$$\begin{array}{ccccc} \mathrm{USp}(4) & \longleftarrow & \mathrm{SU}(2)\times\mathrm{SU}(2) & \longleftarrow & \mathrm{SU}(2) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{SO}(5) & \longleftarrow & \mathrm{SO}(4) & \longleftarrow & \mathrm{SO}(3) \end{array}$$

In particular

$$\mathrm{USp}(4) \cong \mathrm{Spin}(5) \quad \mathrm{SU}(2)\times\mathrm{SU}(2) \cong \mathrm{Spin}(4) \quad \mathrm{SU}(2) \cong \mathrm{Spin}(3)$$

where each $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$.

Explicit example

Let $C : y^2+(x^3+1)y = x^2+x$ be the genus-2 curve in the class 249.a of the LMFDB, and let $\mathrm{Kum}(J)$ be the Kummer surface associated to its Jacobian J. The normalized characteristic polynomials on the transcendental part of $H^2_{et}(\mathcal{K}, \mathbb{Q}_\ell)$ are:

$$\begin{array}{c} x^4 + s_1x^3 + s_2x^2 + s_2x + 1 \\ \hline x^4 + \frac{16}{7}x^3 + \frac{19}{7}x^2 + \frac{16}{7}x + 1 \\ x^4 + \frac{20}{11}x^3 + \frac{19}{11}x^2 + \frac{20}{11}x + 1 \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ x^4 + \frac{32}{31}x^3 + \frac{11}{31}x^2 + \frac{32}{31}x + 1 \\ \vdots \quad\quad \vdots \quad\quad \vdots \end{array}$$

In terms of the characters $(1, s_1, s_2)$ at the initial 43 nonspecial primes up to 251 we obtain the inner product matrix:

$$\begin{pmatrix} 0.999999 & -0.073111 & 0.004920 \\ -0.073111 & 0.833634 & 0.028870 \\ 0.004920 & 0.028870 & 0.904753 \end{pmatrix}$$

consistent with expectations for irreducibility of the characters s_1 and s_2 on $\mathrm{SO}(5)$.

Perspective

Current work focuses on the study of the Sato–Tate groups of Kummer surfaces using the character-theoretic approach. A first step is to understand the identity component ST^0, taking into account that there are six possible connected Sato–Tate groups for abelian surfaces, namely

$$\mathrm{USp}(4), \quad \mathrm{SU}(2)\times\mathrm{SU}(2), \quad \mathrm{SO}(2)\times\mathrm{SO}(2),$$
$$\Delta(\mathrm{SU}(2)), \quad \Delta(\mathrm{SO}(2)), \quad \mathrm{SU}(2)\times\mathrm{SO}(2).$$

Using the exceptional isomorphisms between Lie groups of small rank, the orthogonal Sato–Tate groups of Kummer surfaces can be described as quotients of the corresponding symplectic representations of abelian surfaces. This provides a natural framework for investigating the structure of their Sato–Tate groups and associated character distributions.