On the extension of Triangular Diophantine pairs

Alan Filipin

Faculty of Civil Engineering, University of Zagreb

Introduction

The study of Diophantine sets originates in the classical work of Diophantus, who exhibited sets of rational numbers with the property that the product of any two distinct elements, increased by 1, is a perfect square. This problem has since inspired numerous generalizations, particularly in the direction of replacing squares by other figurate numbers. One natural generalization considers triangular numbers $\Delta_k = \frac{k(k+1)}{2}$, $k \in \mathbb{N}$.

Definition *Let n be an integer. A set $\{a_1, \ldots, a_m\}$ of m distinct positive integers such that $a_i a_j + n$ is a triangular number, for $1 \le i < j \le m$, is called a Triangular Diophantine m-tuple with the property $D(n)$ or simply a Triangular $D(n)$-m-tuple.*

In analogy with the classical case, a central problem is to determine the maximal possible size of such sets and to understand the conditions under which a given tuple can be extended. Several authors have shown that specific pairs of the form $\{n, 2n\}$ cannot be extended to quadruples for certain values of n. These results rely on transforming the problem into systems of Pell-like equations and, in more difficult cases, to determining integral points on elliptic curves. Filipin and Szalay [3] studied the extensibility of the Triangular $D(1)$-pair $\{1, 2\}$, and proved that it cannot be extended to a Triangular $D(1)$-quadruple $\{1, 2, c, d\}$. Filipin, Jurasić, and Szalay [1] proved that the Triangular $D(3)$-pair $\{3, 6\}$ cannot be extended to a Triangular $D(3)$-quadruple $\{3, 6, c, d\}$.

Also, Filipin, Jurasić, and Togbé [2] proved that the Triangular $D(n)$-pair $\{n, 2n\}$, for $n = 10$ and $n = 28$, cannot be extended to a quadruple. The problem described in [2] does not belong to the generalization we are dealing with here, because $10 = 2 \cdot 2^2 + 2$ and $28 = 2 \cdot 4^2 - 4$, but $4 \cdot 2 + 1 = 9$ and $4 \cdot 4 - 1 = 15$ are not primes and k is even. Motivation for this work was to generalize such results and to study the extension of Triangular $D(n)$-pair of the form $\{n, 2n\}$. However, such problem is not straight-forward to solve. The main problem is finding fundamental solutions of Pell-like equations obtained by transforming our problem to solving the system of such equations. More precisely, the main problem was searching of integral points on elliptic curves associated to the problem as we will explain below. Thus, we have to restrict ourselves to the set of the pairs (k, ε), suitable for solving, for now. We consider the family

$$n = 2k^2 + \varepsilon k,$$

where k is an odd positive integer and $\varepsilon \in \{-1, 1\}$, under the additional condition that $4k + \varepsilon$ is prime. This condition ensures a favorable arithmetic structure that allows us to treat several cases simultaneously. Our goal is to investigate whether the corresponding triangular $D(n)$-pair $\{n, 2n\}$ can be extended to a quadruple. The main result of our work is the following theorem:

The result

Let k be an odd positive integer with $4k + \varepsilon$ prime, where $\varepsilon \in \{-1, 1\}$. There does not exist a Triangular $D(2k^2 + \varepsilon k)$-quadruple of the form $\{2k^2 + \varepsilon k, 2(2k^2 + \varepsilon k), c, d\}$, where

$$(k, \varepsilon) \in S := \{(3, -1), (3, 1), (5, -1), (7, 1), (9, 1), (11, -1), (13, 1), (27, 1)\}. \tag{1}$$

The proof

The proof is based on reducing the problem to a system of generalized Pell equations and analyzing their solutions via recurrence sequences. A key step is the connection between the parameters involved and generalized pentagonal numbers, which provides a structural constraint on possible extensions. In addition, the problem leads naturally to the study of integral points on certain elliptic curves, whose analysis imposes further restrictions. Although our result is presently limited to a finite set of parameters, the method developed here applies uniformly across the family and highlights the main obstacles to a complete solution. This leads us to formulate the following conjecture.

Conjecture *Let k be an odd positive integer with $4k + \varepsilon$ prime, where $\varepsilon \in \{-1, 1\}$. Then, there does not exist a Triangular $D(2k^2 + \varepsilon k)$-quadruple of the form $\{2k^2 + \varepsilon k, 2(2k^2 + \varepsilon k), c, d\}$.*

This is a generalization of the cases where $k = 1$, observed in [3] for $\varepsilon = -1$ and in [1] for $\varepsilon = 1$.

For the proof of the main result we observe the general form of the problem and, firstly, we find the sequence $(c_l)_{l \in \mathbb{N}}$ of possible values of c such that $\{2k^2 + \varepsilon k, 2(2k^2 + \varepsilon k), c\}$ is a Triangular $D(2k^2 + \varepsilon k)$-triple. After that, we derive the associated system of generalized Pell equations and describe the structure of their solutions. Next we establish auxiliary properties of the parameters and their relation to figurate numbers. Then, we analyze the fundamental solutions. We complete the proof by combining congruence arguments, properties of the associated recurrence sequences, and estimates arising from linear forms in logarithms to exclude all possible extensions.

In some part of the proof, the authors utilized *ChatGPT-5.2 Thinking* to assist in eliminating all the cases that appear in finding the fundamental solutions of Pell-like equations. It has saved us lot of time, and helped us in some places where we got stuck. However, all resulting solutions were corrected and subsequently verified manually by the authors.

Bibliography

[1] A. Filipin, A. Jurasić and L. Szalay, Extensions of the Triangular Diophantine $D(3)$-pair $\{3, 6\}$, *Math. Slovaca* **75** (2025), 775-790.

[2] A. Filipin, A. Jurasić and A. Togbé, A note on the extension of some Triangular Diophantine sets, *An. Stiint. Univ. "Ovidius" Constanta Ser. Mat.*, to appear.

[3] A. Filipin and L. Szalay, Triangular Diophantine tuples from $\{1, 2\}$, *Rad Hrvat. Akad. Znan. Umjet. Mat. Znan.* **27** (2023), 55–70.

Acknowledgements

This is joint work with Y. Fujita and A. Jurasić. The author is supported by Croatian Science Fund, grant number IP-2022-10-5008. The first author was also supported from the project "Implementation of cutting-edge research and its application as part of the Scientific Center of Excellence for Quantum and Complex Systems, and Representations of Lie Algebras", Grant No. PK.1.1.10.0004, co-financed by the European Union through the European Regional Development Fund - Competitiveness and Cohesion Programme 2021-2027. The part of the work that was done by the author was finished during his visit to the Max Planck Institute for Mathematics in Bonn.

Contact: Alan Filipin, **alan.filipin@grad.unizg.hr**