Algorithmic Obstructions to Morphisms of Theta Lifts and Relative Trace Formula Families

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"It is often helpful to know what is true before proving a theorem." — B.J. Birch

Introduction, Definitions and Goals

Our goal is to study when two theta lifts are isomorphic and to construct families of matching reductive dual pairs, subgroups, and automorphic characters.

Rather than attacking this global problem directly, we study computable local obstructions. For a dual reductive pair $H \times G$ with H orthogonal and G symplectic or metaplectic, characters of H admit an explicit description in terms of Hilbert symbols and local field invariants.

We associate to each local theta lift a collection of fingerprints, including:

- central characters,
- Jacquet-module coinvariants,
- stabilizer data,
- first-occurrence indices in Witt towers.

Comparing these invariants yields efficient tests that rule out possible isomorphisms of theta lifts. Similar local conditions arise in the work of Mao and Rallis on relative trace formulae, suggesting a common underlying matching principle.

The long-term objective is to develop algorithms that generate admissible matching data and predict morphisms of theta lifts from local representation-theoretic invariants.

Examples of Local Matchings and Their Consequences

(1) **Kudla's Matching Principle**

For suitable quadratic spaces $(V,q),(V',q')$, one can construct matching Schwartz functions $\phi \in S(V(\mathbb{A})), \phi' \in S(V'(\mathbb{A}))$ whose images under the local Weil representation ω_{V_v,ψ_v} coincide where $\psi_v : F_v \to \mathbb{C}$ is a fixed additive character.

Applying the Siegel–Weil formula twice gives

$$I_V(g,\phi) = E(g,s_0,\Phi) = I_{V'}(g,\phi'),$$

and hence $I_V(g,\phi) = I_{V'}(g,\phi')$.

Consequence. Matching local data produces arithmetic invariants of Shimura varieties.

(2) **Morphisms of Theta Lifts**

Work of Snitz shows that isomorphic theta lifts satisfy analogous averaging and weight formulas.

The key local ingredients are:

- central characters, Jacquet-module coinvariants, local theta-correspondence data.

This suggests that morphisms of theta lifts are controlled by computable local invariants.

Consequence. Matching local data produces identities between theta series, representation numbers, genera, spinor genera.

(3) **Families of Relative Trace Formulae**

Mao–Rallis construct families of relative trace formulae using local conditions on coinvariants of the Weil representation.

These conditions are strikingly similar to those arising in theta correspondence:

$$Loc_{MR} \approx Loc_{\Theta}.$$

Both involve the same dual pairs, Weil representations, central characters, and Jacquet-module data.

$$\boxed{\text{Morphisms of Theta Lifts} \Longleftrightarrow \text{Families of Relative Trace Formulae}}$$

Morphisms of Theta Lifts

Fix local data $\alpha = ((q_1,V_1),\ \xi_1 : H_{1,v} \to \mathbb{C};\ (q_2,V_2),\ \xi_2 : H_{2,v} \to \mathbb{C})$.

We ask whether

$$\otimes'_v \pi_{1,v} = \Theta(\xi_1,V_1) \cong \Theta(\xi_2,V_2) = \otimes'_v \pi_{2,v}.$$

Here $\pi_{i,v}$ denote the local theta lifts for $i = 1,2$. We study invariants of such pairs of representations of the symplectic (or metaplectic) group G_v, acting on Schwartz spaces.

This comparison is governed by local compatibility conditions

$$\mathrm{Loc}_{\Theta}(\alpha),$$

which encode constraints on central characters, local Fourier data, and Jacquet-module structure.

Goal. Characterize admissible local data α that force isomorphisms of theta lifts

Morphisms of Theta Lifts vs Relative Trace Formulae

Conjecture 1. *If $\alpha \in Loc_{\Theta} = Loc_{MR}$, then*

$$\otimes'_v \pi_{1,v} = \Theta(\xi_1,V_1) \cong \Theta(\xi_2,V_2) = \otimes'_v \pi_{2,v}$$

if and only if there exists a matching relative trace identity

$$I_H(f; H_1,\xi_1,H_2,\xi_2) = I_G(\tilde{f}; N,\psi_\beta,\overline{N},\psi_\beta^{-1}).$$

Here $f \leftrightarrow \tilde{f}$ are matching test functions.

Algorithmic Theta-Lift Obstruction Machine

Fix a non-archimedean local field F_v. We compare two data sets $(H,q,\xi),(H',q',\xi')$. All local representation-theoretic invariants depend only on the associated quadratic forms and characters.

Theta fingerprint. To each pair $((q,V),\xi : H \to \mathbb{C})$ we attach:

$$\mathcal{F}(q,\xi) = (\omega_v,\ r_v,\ J_v),$$

where:

- ω_v: central character of the local theta lift π_v, explicitly determined by Hilbert symbols of q and ξ,
- r_v: first occurrence index in the Witt tower, read off from the Witt decomposition of q,
- J_v: Jacquet-module data of the Weil representation, characterized by the condition that ξ restricts trivially to a stabilizer subgroup in the orthogonal group. This reduces to checking the triviality of certain Hilbert symbols and whether the quadratic form represents an element in the square class of $\lambda \in F_v$. As $F_v^\times/(F_v^\times)^2$ is finite, this yields an algorithmic procedure.

Obstruction principle. If the local theta lifts are isomorphic, i.e. $\pi_v \cong \pi_v$, then necessarily $\mathcal{F}(q,\xi) = \mathcal{F}(q',\xi')$.

Algorithmic Theta-Lift Obstruction Machine

Definition 1 (CC - Central character condition). *A quadruple*

$$\alpha_v = ((q_v,V_v),\xi_v,(q'_v,V'_v),\xi'_v)$$

*is said to satisfy the **central character condition** if π_v and π'_v have the same central character.*

We define the **local Fourier coefficient** of π_v to be the co-invariant space $\left(\left(\omega_{V_v,\psi_v}\right)_{H_v,\xi_v}\right)_{N_v,\psi_\beta}$.

Definition 2 (FC - Fourier coefficient condition). *A quadruple $\alpha_v = ((q_v,V_v),\xi_v,(q'_v,V'_v),\xi'_v)$ is said to satisfy the **Fourier coefficient condition** if and only if*

$$\left(\left(\omega_{V_v,\psi_v}\right)_{H_v,\xi_v}\right)_{N_v,\psi_\beta} = 0 \iff \left(\left(\omega_{V'_v,\psi_v}\right)_{H'_v,\xi' v}\right)_{N_v,\psi_\beta} = 0$$

for every $\beta \in Sym(\mathbb{F})$.

Definition 3 (Locally admissible quadruple). *A quadruple $\alpha_v = ((q_v,V_v),\xi_v,(q'_v,V'_v),\xi'_v)$ is called **locally admissible** if and only if it satisfies the central character (CC) and the Fourier coefficient (FC) conditions.*

Fix a non-archimedean local field F_v.

A local admissible quadruple *is*

$$\alpha_v = ((q,\xi),(q',\xi')),$$

is a quadruple that satisfies the central character and Fourier coefficient conditions.

Admissible quadruple fingerprints.

Definition 4 (Theta fingerprint of an admissible quadruple). *To a local admissible quadruple α_v we attach*

$$\mathcal{F}(\alpha_v) = (\mathrm{CC}(\alpha_v),\ \mathrm{FC}(\alpha_v)),$$

where:

- ***(CC)** Central character data determined by $\mathrm{HP}(q), \mathrm{HP}(q')$ via the spinor norm and Hilbert symbols,*
- ***(FC)** Fourier/Jacquet data determined by degeneracy of quadratic spaces encoded in $\mathrm{HP}(q), \mathrm{HP}(q')$.*

In the following F is a local field and all computations are done locally unless stated otherwise.

Algorithm I: InKernel(α, h).

(1) **Input:**
- A local datum $\alpha = ((q,V),\xi)$, where $H(F) = O(V,q)$ and $\xi : H(F) \longrightarrow \{\pm 1\}$ is the character $\xi(h) = (\mathrm{SN}(h),\lambda)_F$.
- An element $h \in H(F)$.

(2) By the Cartan–Dieudonn'e theorem, write h as a product of reflections $h = r_{v_1} \cdots r_{v_m}$, where $r_v(x) = x - \frac{2B(x,v)}{q(v)} v$.

(3) Compute $a := q(v_1) \cdots q(v_m)$.
Then $\mathrm{SN}(h) = a(F^\times)^2$.

(4) Test whether $(a,\lambda)_F = 1$.
Equivalently, determine whether $a \in N_{F(\sqrt{\lambda})/F}\left(F(\sqrt{\lambda})^\times\right)$.

(5) If the condition in Step (4) holds, return `True`; otherwise return `False`.

Output:

$$\texttt{True} \iff h \in \ker(\xi) \iff \xi(h) = 1.$$

Algorithm II: TestTrivialityOnStabilizer(α, $(v_1,\dots,v_k)$).

(1) **Input:**
- A local datum $\alpha = ((V,q),\xi)$ where $H(F) = O(V,q)$ and
$$\xi(g) = (\mathrm{SN}(g),\lambda)_F.$$
- A tuple $= (v_1,\dots,v_k) \in V^k$ of linearly independent vectors.

(2) Set $U = \mathrm{span}(v_1,\dots,v_k)$ and $W = U^\perp$.

(3) **Structural reduction:** every element of the stabilizer
$$g \in H^{(v_1,\dots,v_k)} \text{ satisfies } g \in O(V,q) : gv_i = v_i\ \forall 1 \le i \le k$$
acts trivially on U and preserves W. Hence restriction induces a map
$$H^{(v_1,\dots,v_k)} \cong O(W,q|_W).$$

(4) **Key reduction:** $\xi|_{H^{\mathbf{v}}} \equiv 1$ if and only if for every $h \in H^{\mathbf{v}}$ we have $InKernel(h) = true$.

(5) Choose a basis of W and represent elements of $O(W,q|_W)$ as matrices in this basis.

(6) **Finite reduction principle (crucial step):** Although $O(W,q|_W)$ is infinite, it is generated by reflections. By the Cartan–Dieudonné theorem, every element of $O(W)$ is a product of reflections in vectors of W. Choose an orthogonal basis $(w_1,\dots,w_m)$ of W. Then the reflections
$$r_{w_1},\dots,r_{w_m}$$
generate $O(W,q|_W)$ up to finite products. Hence there exists a finite generating set
$$\mathcal{G} \subset O(W,q|_W)$$
constructed explicitly from $(w_1,\dots,w_m)$ such that every element of $O(W)$ is a finite product of elements of $\mathcal{G}$.

(7) **Finite test procedure:** To test whether ξ is trivial on $H^{(v_1,\dots,v_k)}$, it suffices to check:
$$\textsc{InKernel}(\alpha,g) = \texttt{true} \quad \forall g \in \mathcal{G}.$$

(8) **Conclusion step:** If all generators satisfy the kernel condition, then by multiplicativity of the spinor norm and bilinearity of the Hilbert symbol,
$$\xi(h) = 1 \quad \forall h \in H^{(v_1,\dots,v_k)}.$$
Otherwise, the character is nontrivial on the stabilizer.

Output:

$$\texttt{True} \iff H^{(v_1,\dots,v_k)} = 1.$$

Algorithm I: CC-Condition(alpha). - Central Character Condition

(1) **Input:** local data $\alpha_v = ((q,V),\xi),(q',V'),\xi')$ where ξ and ξ' are given by $\lambda,\lambda' \in F_v/F_v^{\times,2}$

(2) By Lemma 4.4 of [1] the central character condition is satisfied if and only if
$$\chi_{V_v}(-I_n)\,\xi_v(-I_n) = \chi_{V'_v}(-I_n)\,\xi'_v(-I_n)$$
This computation depends on computing Hilbert symbols and the determinant of the quadratic spaces. Note that ξ_v, ξ'_v are quadratic characters associated to the Weil representation (see Section 3.3 of [1] for the precise definition).

(3) Return true if equality holds, else return false

Algorithm III: Fourier / Jacquet test.

(1) **Input:** local data $\alpha_v = ((q,V),\xi),(q',V'),\xi')$ where ξ and ξ' are given by $\lambda,\lambda' \in F_v/F_v^{\times,2}$

(2) By Theorem C.13 of [1] the Fourier condition is satisfied if and only if $\lambda \in d(V)d(V') \cdot \square^\times$ and since the square class of a local field is finite this can be tested in a finite number of steps

(3) Return true if equality holds, else return false

Algorithm IV: Local obstruction detector.

(1) **Input:** local data $\alpha_v = ((q,\xi),(q',\xi'))$.

(2) Set $FC \leftarrow CC - Condition(alpha)$

(3) Set $FC \leftarrow CFC - Condition(alpha)$

(4) Return $CC \wedge FC$.

Remark. *It can be shown that for data $\alpha_v = ((q,V),\xi),(q',V'),\xi')$ for nontrivial characters ξ,ξ' the dimensions must satisfy*

$$\dim V, \dim V' \subseteq \{n, n+2\}$$

See Lemma8.11 of [1] for more details.

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