Rank Growth in Anticyclotomic Towers for CM Elliptic Curves

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Abstract

Let E/K be an elliptic curve defined over an imaginary quadratic field K, and assume that E has complex multiplication (CM). This study investigates the behavior of the Mordell-Weil rank of E over the finite layers K_n of the anticyclotomic $\mathbb{Z}_p$-extension of K. Using tools from Iwasawa theory and Selmer groups, we study conditions under which the ranks $\mathrm{rank}_{\mathbb{Z}}\, E\left(K_n\right)$ remain bounded as $n \to \infty$. In particular, we apply a result of Mazur under suitable hypotheses, and illustrate the theory with explicit examples of CM elliptic curves.

Background of the Study

The study of rational points on elliptic curves lies at the heart of modern number theory. By the Mordell-Weil theorem, for a number field K, the group of rational points $E(K)$ is finitely generated:

$$E(K) \cong E(K)_{\mathrm{tors}} \oplus \mathbb{Z}^r$$

where $r = \mathrm{rank}_{\mathbb{Z}}\, E(K)$ is the Mordell-Weil rank. A natural question is how this rank behaves when the base field varies. In particular, given a tower of number fields

$$K \subset K_1 \subset K_2 \subset \cdots,$$

one may ask whether $\mathrm{rank}_{\mathbb{Z}}\, E\left(K_n\right)$ remains bounded or grows indefinitely. This study focuses on the case where:

K is an imaginary quadratic field,

K_∞/K is the anticyclotomic $\mathbb{Z}_p$-extension,

E/K is a CM elliptic curve.

We aim to determine conditions under which the Mordell-Weil rank remains bounded along this tower.

Elliptic Curves and Mordell-Weil Groups
Let E/K be an elliptic curve over a number field. The Mordell-Weil theorem ensures that $E(K)$ is finitely generated. The rank measures the number of independent rational points of infinite order. Understanding how this rank changes under field extensions is a central problem in arithmetic geometry.

Anticyclotomic $\mathbb{Z}_p$-Extensions
Let K be an imaginary quadratic field and p a prime. The anticyclotomic $\mathbb{Z}_p$-extension K_∞/K is the unique infinite extension with Galois group:

$$\mathrm{Gal}\left(K_\infty/K\right) \cong \mathbb{Z}_p,$$

and characterized by its behavior under complex conjugation. We denote by K_n the intermediate fields with:

$$[K_n : K] = p^n.$$

CM Elliptic Curves
An elliptic curve E has complex multiplication (CM) if its endomorphism ring is larger than $\mathbb{Z}$. CM curves possess additional structure that makes them particularly amenable to Iwasawa-theoretic techniques.

Selmer Groups and Iwasawa Theory

Selmer groups
For each n, the p^∞-Selmer group $\mathrm{Sel}_{p^\infty}\left(E/K_n\right)$ is a subgroup of Galois cohomology encoding information about rational points and Tate-Shafarevich groups.

A key inequality relates Selmer groups to ranks:

$$\mathrm{rank}_{\mathbb{Z}}\, E\left(K_n\right) \leq \mathrm{corank}_{\mathbb{Z}_p}\, \mathrm{Sel}_{p^\infty}\left(E/K_n\right).$$

Iwasawa Module Structure
Consider the inverse limit:

$$X = \varprojlim \mathrm{Sel}_{p^\infty}\left(E/K_n\right).$$

This is naturally a module over the Iwasawa algebra:

$$\Lambda = \mathbb{Z}_p\left[\left[\mathrm{Gal}\left(K_\infty/K\right)\right]\right].$$

The structure of X as a Λ-module governs the growth of Selmer groups, and hence the rank.

Mazur's Control Theorem

Theorem (Mazur)**.** *Let K_∞/K be a $\mathbb{Z}_p$-extension, and let E/K be an elliptic curve with good ordinary reduction at all primes above p. Then the natural restriction map*

$$\mathrm{Sel}_{p^\infty}\left(E/K_n\right) \longrightarrow \mathrm{Sel}_{p^\infty}\left(E/K_\infty\right)^{\mathrm{Gal}(F_\infty/F_n)}$$

has finite kernel and cokernel bounded independently of n. Hence the Selmer groups over the finite layers K_n are controlled by the Selmer group over the infinite extension K_∞.

Recall that a Λ-module X is cotorsion if it is "small" in the sense that it has no free Λ-part. Equivalently, its structure looks like:

$$X \sim \bigoplus_i \Lambda/\left(f_i(T)^{n_i}\right).$$

Under Mazur's hypotheses, the Selmer group $\mathrm{Sel}_{p^\infty}\left(E/K_\infty\right)$ forms a cotorsion module over Λ, that is, it has no free component that could drive growth. As a result, its size, and thus the Mordell-Weil rank, remains uniformly bounded across the tower.

Main Result

We now state the main result, which follows from Mazur's Control Theorem.

Theorem. *Let K be an imaginary quadratic field, let E/K be an elliptic curve with complex multiplication, and let p be a prime. Let K_∞/K be the anticyclotomic $\mathbb{Z}_p$-extension of K, and let K_n denote its finite layers. Suppose that:*

(i) E has good ordinary reduction at every prime of K above p,

(ii) $E(K)$ is finite,

(iii) the p-primary subgroup of the Tate-Shafarevich group $Ш(E/K)\left[p^\infty\right]$ is finite.

Then the Mordell-Weil ranks

$$\mathrm{rank}_{\mathbb{Z}}\, E\left(K_n\right)$$

remain bounded as $n \to \infty$.

Examples

1. Let $K = \mathbb{Q}(i)$ and $E : y^2 = x^3 - x$ (which has CM by $\mathbb{Z}[i]$). For $p = 5$, the elliptic curve has good ordinary reduction. Over the anticyclotomic $\mathbb{Z}_5$-extension K_∞/K with layers K_n, we have $\mathrm{rank}_{\mathbb{Z}}\, E\left(K_n\right) = 0, \forall n$.
2. Let $K = \mathbb{Q}(\sqrt{-7})$ and $E : y^2 + xy = x^3 - x^2 - 2x - 1$ (which has CM by the ring of integers of K). For $p = 3$, we have $\mathrm{rank}_{\mathbb{Z}}\, E(K) = 1$, and $\mathrm{rank}_{\mathbb{Z}}\, E\left(K_n\right)$ remains bounded as across the anticyclotomic tower.

Note that in the anticyclotomic tower for CM curves, rank growth is tightly controlled, even in cases where the base curve already has positive rank.

References

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