Chabauty–Coleman on Symmetric Squares of Picard Curves

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Classical Chabauty–Coleman Bound on Curves

Theorem(Coleman, 1985): Let $C/\mathbb{Q}$ be a smooth, projective, geometrically irreducible curve of genus $g \geq 2$ with Jacobian J_C and p a prime of good reduction for C. We assume:

- $p > 2g$,
- *Dimension inequality*: $\operatorname{rk} J_C(\mathbb{Q}) + 1 \leq g$,

$$\#C(\mathbb{Q}) \leq \#C(\mathbb{F}_p) + 2g - 2.$$

Explicit Computation:

- Fix a base point $P_0 \in C(\mathbb{Q})$.
- let $\overline{J_C(\mathbb{Q})}$ be the p-adic closure of $J_C(\mathbb{Q})$
- Ann$(C(\mathbb{Q})) = \{\omega \in H^0(C_{\mathbb{Q}_p}, \Omega^1) : \int_D \omega = 0 \ \ \forall D \in J_C(\mathbb{Q})$ and $\omega \neq 0\}$.

then $C(\mathbb{Q})$ is a subset of

$$C(\mathbb{Q}_p)_1 = \{P \in C(\mathbb{Q}_p) : \int_{P_0}^{P} \omega = 0 \ \ \forall \omega \in \text{Ann}(C(\mathbb{Q}))\}.$$

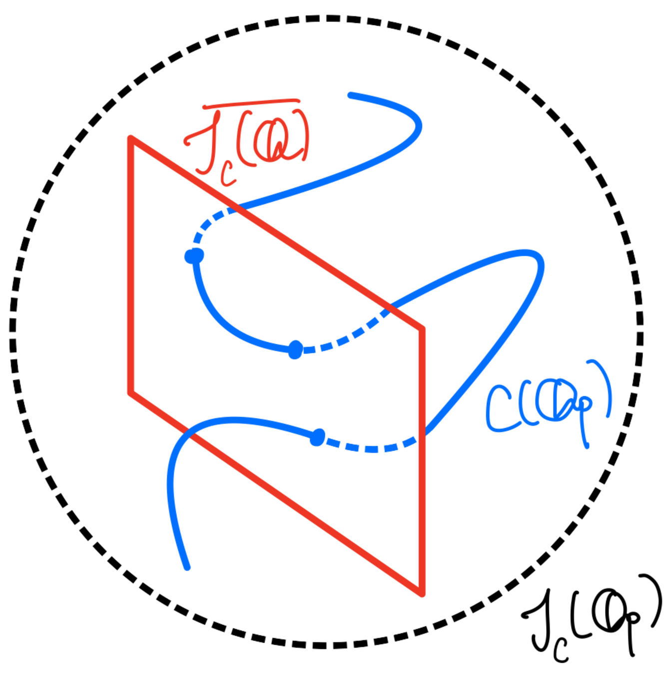


Figure 1. positive codimensional $\overline{J_c(\mathbb{Q})}$ intersects the 1-dimensional submanifold $C(\mathbb{Q}_p)$ of $J_C(\mathbb{Q}_p)$ in only finitely many points

Chabauty–Coleman beyond curves

What about a Chabauty-Coleman bound for X a higher dimensional variety? With the assumptions:

- X is of general type, contained in an abelian variety A and $X(\mathbb{Q})$ is finite.
- *Dimension inequality*: $\operatorname{rk} A(\mathbb{Q}) + \dim X \leq \dim A$.

Chabauty–Coleman Bound on Surfaces

Theorem(Caro-Pasten 2023): Let X be a smooth, geometrically irreducible, projective surface of general type in an abelian variety A of dimension 3 over $\mathbb{Q}$. Let $c_1^2(X)$ denote the self-intersection of the canonical divisor K_X of X,

We assume:

- $p > (\frac{128}{9}) \cdot (c_1^2(X))^2$ and is a prime of good reduction for X and A.
- $\operatorname{rk} A(\mathbb{Q}) \leq 1$ (Dimension inequality),
- X contains no elliptic curves over $\mathbb{F}_p^{\text{alg}}$,

Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + \frac{p-1}{p-2} \cdot \left(p + 4p^{1/2} + 5\right) \cdot c_1^2(X)$$

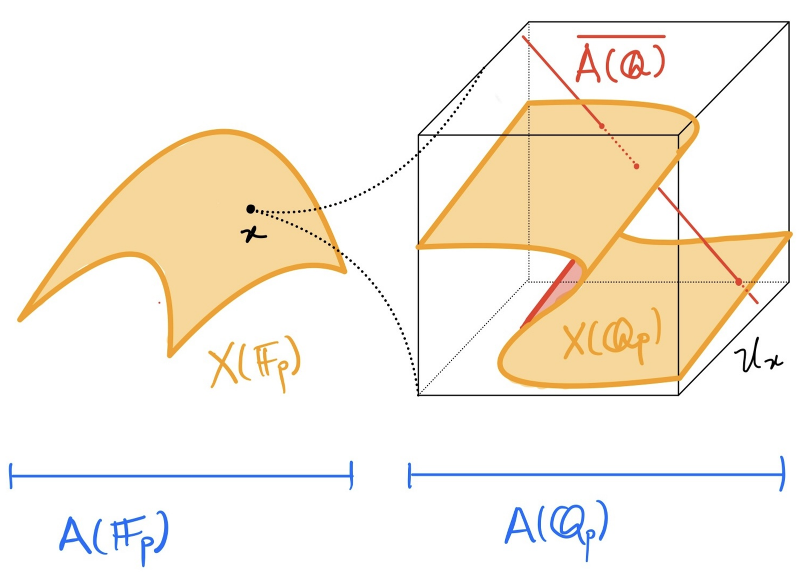
Idea of the Proof

- $X(\mathbb{Q}) \subset X(\mathbb{Q}_p) \cap \overline{A(\mathbb{Q})}$, where $\overline{A(\mathbb{Q})}$ is the p-adic closure.
- Let red : $A(\mathbb{Q}_p) \to A(\mathbb{F}_p)$. For each $x \in X(\mathbb{F}_p)$ it is possible to bound

$$\#X(\mathbb{Q}_p) \cap \overline{A(\mathbb{Q})} \cap U_x$$

  where $U_x = \text{red}^{-1}(x)$.
- Adding over $X(\mathbb{F}_p)$

$$\#X(\mathbb{Q}_p) \cap \overline{A(\mathbb{Q})} = \sum_{x \in X(\mathbb{F}_p)} \#X(\mathbb{Q}_p) \cap \overline{A(\mathbb{Q})} \cap U_x$$



Key Step: $\#X(\mathbb{Q}_p) \cap \overline{A(\mathbb{Q})} \cap U_x \leq 1 + \frac{p-1}{p-2} \cdot$ **m(x)**

m(x) : Depends on a certain divisor $D :=$ div$(\tilde{\omega}_1 \wedge \tilde{\omega}_2)$ in $\operatorname{Div} X_{\overline{\mathbb{F}}_p}$ where $\{\omega_1, \omega_2\}$ are certain differentials in $H^0(A, \Omega^1_{A/\mathbb{Q}_p})$ and $\tilde{\omega}_i = \text{red}(\omega_i)|_X$.

Application of Caro-Pasten Method

Let $C/\mathbb{Q}$ be a smooth, geometrically irreducible, projective curve of genus 3 such that $\operatorname{rk} J_C(\mathbb{Q}) \leq 1$. Then $C^{(2)} := C^2/S_2$ is a smooth surface of general type with an embedding into J_C. Let W_2 be the image of $C^{(2)}$ in J_C, a closed subvariety of J_C.

Theorem(Caro-Pasten 2023): Let $p \geq 521$ be a prime of good reduction for C. Suppose W_2 does not contain elliptic curves over $\mathbb{F}_p^{alg}$. Then $W_2(\mathbb{Q})$ is finite and

$$\#W_2(\mathbb{Q}) \leq \#W_2(\mathbb{F}_p) + 6 \cdot \frac{p-1}{p-2} \cdot \left(p + 4p^{1/2} + 5\right)$$

Chabauty–Coleman on the Symmetric Square of Hyperelliptic Curves

Let $C/\mathbb{Q}$ be a hyperelliptic curve of genus 3 such that $\operatorname{rk} J_C(\mathbb{Q}) \leq 1$. Then $C^{(2)}$ is a blow-up of the surface $W_2 := C + C \subset J_C$.

Theorem(Balakrishnan-Caro 2025): Let $p \geq 11$ be a prime of good reduction for C. Suppose that W_2 does not contain elliptic curves over $\mathbb{F}_p^{\text{alg}}$. Then

$$\#W_2(\mathbb{Q}) \leq \#W_2(\mathbb{F}_p) + 2p + 12\sqrt{p} + 7$$

Explicit Computation((Balakrishnan-Caro): Implemented an algorithm in magma to compute upper bound on $m(x)$ for each $x \in W_2(\mathbb{F}_p)$.

Chabauty–Coleman on the Symmetric Square of Picard Curves

A *Picard curve* $\mathcal{C}$ over $\mathbb{Q}$ is a smooth projective non-hyperelliptic curve of genus 3 with an affine equation $y^3 = f(x)$ where f is a quartic polynomial.

Let $\mathcal{C}/\mathbb{Q}$ be a Picard curve such that $\operatorname{rk} J_{\mathcal{C}}(\mathbb{Q}) \leq 1$ and without the existence of a degree 2 map $\mathcal{C} \to D$ where D is a curve with infinitely many rational points.

- Fix the base point $P_\infty = (0, 1, 0) \in \mathcal{C}(\mathbb{Q})$.

for all $P, Q \in \mathcal{C}(\mathbb{Q})$

- $j(P) := (P, P_0)$,
- $\text{sym}(P, Q) := \{P, Q\}$
- $i(P) := [P - P_0]$
- $\psi : \mathcal{C}^{(2)} \hookrightarrow J_{\mathcal{C}}$ via $\psi(\{P, Q\}) := [P + Q - 2P_0]$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "j"] \arrow[rrd, "i"'] & \mathcal{C}^2 \arrow[r, "\text{sym}"] & \mathcal{C}^{(2)} \arrow[d, "\psi"] \\ & & J_{\mathcal{C}} \end{tikzcd}$$

- Hence, $W_2 := \operatorname{Im}(\psi) \cong \mathcal{C}^{(2)}$.

Goal

The goal is to give a refined upper bound on

$$\mathcal{C}^{(2)}(\mathbb{Q}) = W_2(\mathbb{Q}) = \{\text{unordered pairs of rational points \& quadratic points}\}$$

- Sharpen the bound $\#\mathcal{C}^{(2)}(\mathbb{Q}_p) \cap \overline{J_{\mathcal{C}}(\mathbb{Q})} \cap U_x \leq 1 + \frac{p-1}{p-2} \cdot$ **m(x)**.
- lower the bound on the prime p since $\mathcal{O}(\#W_2(\mathbb{F}_p)) = p^2$.

Achieved by focusing on **m(x)** $\rightsquigarrow$ so on the divisor $D = $ div$(\tilde{\omega}_1 \wedge \tilde{\omega}_2)$.

Main Result

Theorem(Caro–PM): Let **$p \geq 19$** be a prime of good reduction for $\mathcal{C}$. Suppose that W_2 does not contain elliptic curves over $\mathbb{F}_p^{\text{alg}}$. Then $W_2(\mathbb{Q})$ is finite and

$$\#W_2(\mathbb{Q}) \leq \#W_2(\mathbb{F}_p) + 2p + 16\sqrt{p} + 20$$

Strategy

- Construct the divisor $D = $ div$(\tilde{\omega}_1 \wedge \tilde{\omega}_2)$ where $\{\omega_1, \omega_2\} \in$ Ann$(\mathcal{C}(\mathbb{Q}))$ independent differentials.
- We find $D = \psi_* \text{sym}_*(Z' - \Delta_C)$ where

$$Z' = \mathbb{V}(\beta_{12}(x_2 - x_1) + \beta_{23}(x_1 y_2 - x_2 y_1) + \beta_{13}(y_2 - y_1)) \subset \mathcal{C}^2/\mathbb{F}_p.$$

- The curve $Z = Z' - \Delta_C$ has at most four irreducible components. Generically, Z is irreducible with arithmetic genus 19.
- If $z \notin Z$, then $m(z) = 0$. So only need $z \in Z$.
- Use Riemann-Roch theorem for the components of Z to get a general upper bound.

We have magma code to compute upper bounds of $m(x)$, on each U_x, when Z is irreducible.

Algorithm for an improved upper bound via $m(z)$ (Implemented in Magma)

Input: Rank 1 Picard Curve of the form $y^3 = f(x)$ where $f \in \mathbb{Z}[x]$.

1. Step 1:
   - Compute generators of a finite index subgroup G of the Jacobian $J(\mathbb{Q})$.
   - Choose the lowest prime of good reduction p such that red$_p J(\mathbb{Q}) = $ red$_p G$.
2. Step 2:
   - Compute independent annihilating differentials ω_1, ω_2.
   - Reduce ω_1, ω_2 modulo p to find the equation of $Z = $ supp(D).
   - Check whether Z is irreducible.
3. Step 3:
   - Compute the set red$_p J(\mathbb{Q}) \cap W_2(\mathbb{F}_p)$.
4. Step 4:
   - For $z \in Z$, find an upper bound on $m(z)$, if $z \notin Z$, then $m(z) = 0$.
   - For each $z \in W_2(\mathbb{F}_p)$ we have $\#W_2(\mathbb{Q}_p) \cap \bar{J}(\mathbb{Q}) \cap U_z \leq m(z) + 1$ or $m(z) + 2$ quadratic points in $W_2(\mathbb{Q})$.
5. Step 5: Adding over all points in red$_p J(\mathbb{Q}) \cap W_2(\mathbb{F}_p)$ to deduce an upper bound for $\#W_2(\mathbb{Q})$.

Example *Input:* $\mathcal{C} : y^3 = x^4 + 3x^3 - 12x^2 + 9x$.

- $J_{\mathcal{C}}(\mathbb{Q}) \cong \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Then $G = \langle P, Q \rangle \subset J(\mathbb{Q})$ is a subgroup of finite index where $P := [(1,1) - \infty]$ has infinite order and $Q = [(0,0) - \infty]$ generates $J(\mathbb{Q})[3]$.
- $p = 5$ a prime of good reduction of $\mathcal{C}$, and

$$Z' : \mathbb{V}(x_2 z_1 + 4x_1 z_2 + 4x_1 y_2 + y_1 x_2 + 4z_1 y_2 + z_2 y_1) \subset \mathcal{C}^2/\mathbb{F}_5$$

  where Z is irreducible but singular.
- red$_5 W_2(\mathbb{Q}) \subset S :=$ red$_5 J(\mathbb{Q}) \cap W_2(\mathbb{F}_5)$ has 7 points and

$$S = \{0_J, [\overline{(1,1)} - \infty], [2\overline{(1,1)} - 2\infty], [\overline{(0,0)} - \infty], [2\overline{(0,0)} - \infty], [\overline{(1,\zeta_3)} + \overline{(1,\zeta_3^2)} - 2\infty], [\overline{(0,0)} + \overline{(1,1)} - 2\infty]\}$$

- $S \cap Z = \{[2\overline{(1,1)} - 2\infty], [\overline{(0,0)} + \overline{(1,1)} - 2\infty]\}$.
- $m([\overline{(0,0)} + \overline{(1,1)} - 2\infty]) \leq 1$, $m([2\overline{(1,1)} - 2\infty]) \leq 2$ and $m(z) = 0$ for all $z \in S \backslash S \cap Z$.
- *Output*: Refined Upper Bound: $\#W_2(\mathbb{Q}) \leq 5 + 2 + 3 = 10$, with a lower bound equals 8 since

$$W_2(\mathbb{Q}) \supset \{0_J, [(1,1) - \infty], [2(1,1) - 2\infty], [(0,0) - \infty], [2(0,0) - 2\infty], [(1,1) + (0,0) - 2\infty]$$
$$[(1,\zeta_3), +(1,\zeta_3^2) - 2\infty], [(1/2(3\sqrt{5} - 3), 1/2(3\sqrt{5} - 3)) + (1/2(-3\sqrt{5} - 3), 1/2(-3\sqrt{5} - 3)) - 2\infty]\}$$

Main References

[1] Jerson Caro and Hector Pasten. A Chabauty-Coleman bound for surfaces. *Invent. Math.*, 234(3):1197–1250, 2023.

[2] Sachi Hashimoto and Travis Morrison. Chabauty-Coleman computations on rank 1 Picard curves. In *Arithmetic geometry, number theory, and computation*, Simons Symp., pages 485–506. Springer, Cham, [2021] 2021.