Quaternion Orders and Diophantine Equations

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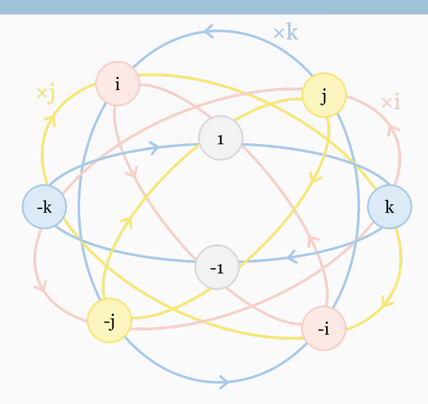
Introduction

One important problem in Number Theory is the study of **Diophantine equations**: algebraic equations with integer coefficients in two or more unknowns, whose integers solutions are of interest. In this work, our focus lies on equations of the form $t^2 - ax^2 - by^2 + abz^2$, since those can be viewed as the norm of an arbitrary integral quaternion, which allows us to use **quaternion orders** to study them. The most famous example of this is Hurwitz's [3] proof of the Four Squares Theorem using a quaternion order inside the 4-dimensional associative division algebra over the rational numbers,

$$\mathbb{H}(\mathbb{Q}) = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k, \text{ where } i^2 = -1 = j^2 \text{ and } ij = k = -ji.$$

Other examples of the use of quaternions in the study of Diophantine Equations include the work of Deutsch [1] and Fitzgerald [2].



Quaternion Algebras

Let $\mathbb{F}$ be a field whose characteristic is not 2 and a and b be non-zero elements in $\mathbb{F}$. We say that $\mathbb{A} := \left(\frac{a,b}{\mathbb{F}}\right)$ is a **quaternion algebra over** $\mathbb{F}$ if it is a central simple 4-dimensional algebra over $\mathbb{F}$ with basis $1, i, j, k$ such that

$$i^2 = a, \quad j^2 = b, \quad ij = k = -ji.$$

The **conjugate** of $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{A}$ is

$$\overline{x} := x_0 - x_1 i - x_2 j - x_3 k.$$

The map $\mathrm{N}\colon \mathbb{A} \to \mathbb{F}$ denotes the **norm** map given by

$$\mathrm{N}(x) = x\overline{x} = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2,$$

and the map $\mathrm{Tr}\colon \mathbb{A} \to \mathbb{F}$ denotes the **trace** map given by

$$\mathrm{Tr}(x) = x + \overline{x} = 2x_0.$$

$\times$	1	i	j	k
1	1	i	j	k
i	i	a	k	aj
j	j	$-k$	b	$-bi$
k	k	$-aj$	bi	$-ab$

Quaternion Orders

Definition. *Let $\mathbb{A}$ be a quaternion algebra over $\mathbb{Q}$. A **(quaternion) order** H of $\mathbb{A}$ is a finitely generated $\mathbb{Z}$-module contained in the algebra, such that H is also a subring of $\mathbb{A}$ and $\mathbb{Q} \otimes H \cong \mathbb{A}$.*

Proposition. *For any order H, we have that $\mathbb{Q} \cap H = \mathbb{Z}$. Moreover the norm and the trace of any element of H is in $\mathbb{Z}$.*

Examples of Quaternion Orders

- In the quaternion algebra $\mathbb{H}(\mathbb{Q}) = \left(\frac{-1,-1}{\mathbb{Q}}\right)$, we have:
  - the ring of **Lipschitz Integers**, denoted by $\mathcal{L}$ and defined as
    $$\mathcal{L} := \mathbb{Z}[1, i, j, k] = \{a + bi + cj + dk \in \mathbb{H}(\mathbb{Q})\colon a, b, c, d \in \mathbb{Z}\}.$$
    $\mathcal{L}$ is an order, but it neither Euclidean for the norm nor a Principal Ideal Domain (PID);
  - the ring of **Hurwitz Integers**, denoted by $\mathcal{H}$. It is characterized by
    $$\mathcal{H} = \mathcal{L} \cup (\mathcal{L} + \omega) = \mathbb{Z}[1, i, j, \omega],$$
    where $\omega = \frac{1+i+j+k}{2}$. It is a maximal order in the above algebra, Euclidean for the norm, and thus a PID.
- In the quaternion algebra $\left(\frac{-2,-5}{\mathbb{Q}}\right)$, we have the orders
  $$\mathcal{L}_{2,5} := \mathbb{Z}[1, i, j, k] \subset \mathcal{H}_{2,5} := \mathbb{Z}\left[1, \frac{2+i-k}{4}, \frac{2+3i+k}{4}, \frac{1+i+j}{2}\right],$$
  where $\mathcal{H}_{2,5}$ is a maximal order and Euclidean for the norm, thus a PID.
- In the quaternion algebra $\left(\frac{-1,-7}{\mathbb{Q}}\right)$, we have the orders
  $$\mathcal{L}_{1,7} := \mathbb{Z}[1, i, j, k] \subset \mathcal{H}_{1,7} := \mathbb{Z}\left[1, i, \frac{1}{2}(1+j), \frac{1}{2}(i+k)\right],$$
  where $\mathcal{H}_{1,7}$ is a maximal order. Although it is not Euclidean for the norm, it is a PID.

Four Squares Theorem

Theorem. *Four Squares Theorem (1770) Any $m \in \mathbb{N}$ can be written as the sum of four integer squares.*

This theorem is equivalent to showing that exits a Lipschitz Integer of norm m, for all $m \in \mathbb{N}$, that is,

$$\gamma = t + xi + yj + zk \in \mathcal{L} \text{ such that } \mathrm{N}(\gamma) = m.$$

The next lemma entails that it is enough to show that there exists an Hurwitz Integer of norm m instead.

Lemma. *If $\gamma \in \mathcal{H} \setminus \mathcal{L}$, then there exists a unit $u \in \mathcal{H}$ such that $u\gamma, \gamma u \in \mathcal{L}$.*

Finally, the existence of an element of norm m in $\mathcal{H}$ is guaranteed by the fact that $\mathcal{H}$ is a PID, since we have the following result.

Theorem 1. *(C., Leite, Machiavelo, Roçadas, Moreira) If H is a PID order in $\mathbb{A}$, then its norm form is universal.*

Metaconjugation

Consider the quadratic form $t^2 - ax^2 - by^2 + abz^2$ and let $\mathbb{A} := \left(\frac{a,b}{\mathbb{Q}}\right)$ be the rational quaternion algebra associated to it.

The problem of determining all the natural numbers n that are represented in $\mathbb{Z}$ by the above form is equivalent to finding all natural numbers n such that there exists an element α in the order of integral quaternions $\mathcal{L}_{a,b} := \mathbb{Z}[1, i, j, k]$ such that $\mathrm{N}(\alpha) = n$.

Our arguments are built upon a process that we call **metaconjugation**, which transforms elements from a PID order into elements of the same norm in $\mathcal{L}_{a,b}$. It works as follows: let $\mathcal{H}_{a,b}$ be a PID order that contains $\mathcal{L}_{a,b}$ and $\gamma \in \mathcal{H}_{a,b}$. If $\sigma, \tau \in \mathcal{H}_{a,b}$ are of equal norm, then $\sigma\gamma\tau^{-1}$ has norm equal to that of γ. Hence, in order to prove that $n \in \mathbb{N}$ is represented by the norm form of $\mathcal{L}_{a,b} = \mathbb{Z}[1, i, j, k]$, it is enough to find $\sigma, \tau \in \mathcal{H}_{a,b}$ such that

$$\sigma\gamma\tau^{-1} \in \mathcal{L}_{a,b},$$

for $\gamma \in \mathcal{H}_{a,b}$ of norm n.

Form $t^2 + 2x^2 + 5y^2 + 10z^2$

Consider the form $t^2 + 2x^2 + 5y^2 + 10z^2$ and its associated order $\mathcal{L}_{2,5} = \mathbb{Z}[1, i, j, k]$ in the quaternion algebra $\left(\frac{-2,-5}{\mathbb{Q}}\right)$, whose canonical basis satisfy $i^2 = -2, j^2 = -5$. Using PARI/GP [4], we proved the following proposition.

Proposition. *Let $\gamma \in \mathcal{H}_{2,5}$. Then there exist $\sigma, \tau \in \mathcal{H}_{2,5}$ of **norm 4**, such that*

$$\sigma\gamma\tau^{-1} \in \mathcal{L}_{2,5}.$$

The above Proposition and Theorem 1 yield the first Main Theorem of this work.

Main Theorem 1. *(C., Leite, Machiavelo, Roçadas, Moreira) The form $t^2 + 2x^2 + 5y^2 + 10z^2$ is universal.*

Form $t^2 + x^2 + 7y^2 + 7z^2$

Consider the form $t^2 + x^2 + 7y^2 + 7z^2$ and its associated order $\mathcal{L}_{1,7} = \mathbb{Z}[1, i, j, k]$, which is an order in the quaternion algebra $\left(\frac{-1,-7}{\mathbb{Q}}\right)$, whose canonical basis satisfy $i^2 = -1, j^2 = -7$. It is easy to see that if $n = 2^\varepsilon \cdot 3 \cdot 7^k$, for some $\varepsilon \in \{0, 1\}$ and $k \in \mathbb{N}_0$, then the above form does not represent n. Using the multiplicative property of the norm, we get the following result.

Theorem 2. *If $t^2 + x^2 + 7y^2 + 7z^2$ **represents all primes different from 3** and **represents** $3p$, **for all odd prime** p **different from 7**, then it represents all n not of the form $n = 2^\varepsilon \cdot 3 \cdot 7^k$, for $\varepsilon \in \{0, 1\}$ and $k \in \mathbb{N}_0$.*

Using PARI/GP [4], we prove the first part of Theorem 2.

Proposition. *If $\gamma \in \mathcal{H}_{1,7} \setminus \mathcal{L}_{1,7}$ is s.t. $\mathrm{N}(\gamma) = p$ for a prime p greater than 3, then there exist $\sigma, \tau \in \mathcal{H}_{1,7}$ of **norm 6** such that*

$$\sigma\gamma\tau^{-1} \in \mathcal{L}_{1,7}.$$

*In particular, **if** p **is a prime different from 3, then** p **is represented** by $t^2 + x^2 + 7y^2 + 7z^2$.*

For the second part of Theorem 2, we show that the form represents $3p$ for an odd prime p if and only if there exists $\gamma \in \mathcal{H}_{1,7} \setminus \mathcal{L}_{1,7}$ such that $\mathrm{N}(\gamma) = p$. We prove this equivalent condition, using the usual conjugation.

Proposition. *For any $\gamma \in \mathcal{L}_{1,7}$ of prime norm p different from 2 and 7, there exists $k \in \mathbb{N}$ such that we have either*

$$\omega^k \gamma \omega^{-k} \text{ or } \bar{\omega}^k \gamma \bar{\omega}^{-k} \in \mathcal{H}_{1,7} \setminus \mathcal{L}_{1,7}.$$

*In other words, there exists $\gamma \in \mathcal{H}_{1,7} \setminus \mathcal{L}_{1,7}$ such that $\mathrm{N}(\gamma) = p$, for **all odd prime** p **different from** 7.*

Corollary. *If p is an odd prime different from 7, then $3p$ is represented by $t^2 + x^2 + 7y^2 + 7z^2$.*

From the above results and Theorem 2 follows the second Main Theorem of this work.

Main Theorem 2. *(C., Leite, Machiavelo, Roçadas, Moreira) The form $t^2 + x^2 + 7y^2 + 7z^2$ represents $n \in \mathbb{N}$ if and only if n is not of the form $n = 2^\varepsilon \cdot 3 \cdot 7^k$, for $\varepsilon \in \{0, 1\}$ and $k \in \mathbb{N}_0$.*

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