Mordell-Weil Lattices and Dense Sphere Packings

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The Sphere Packing Problem

An $n-$dimensional **sphere packing** is a collection of non-overlapping congruent spheres in $\mathbb{R}^n$.
A "dense" sphere packing leaves little free space in $\mathbb{R}^n$. The **packing density** $\Delta_n(\mathscr{P})$ of a packing $\mathscr{P}$ is the ratio between the space covered by the spheres and the total space (if this ratio is well-defined).
For a given dimension n, we can ask: what is the *maximal* packing density $\Delta(n) = \max_{\mathscr{P}} \Delta_n(\mathscr{P})$?
$\Rightarrow$ this is a seriously hard problem!

- We know the solutions only in dimensions $n = 1, 2, 3, 8, 24$
- Very interesting problem not only mathematically and computationally, e.g. applications to error-correcting codes
- For high n, all we have is lower bounds:

$$\Delta(n) \geq \zeta(n) \cdot (n-1) \cdot 2^{-(n-1)} \quad \text{(Ball)}$$

Lattice Packings

A natural framework: spheres are centered in points of an $n-$dim. lattice L and have **packing radius** $\rho = \lambda(L)/2$ where $\lambda(L)$ =shortest vector length in L.

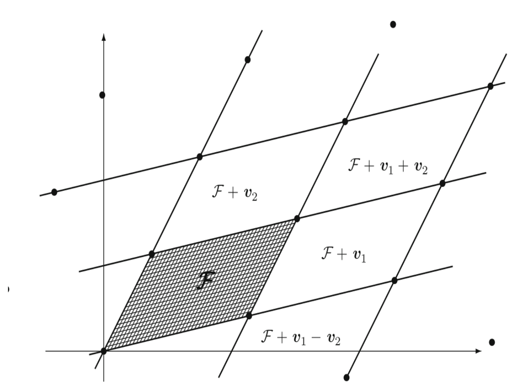
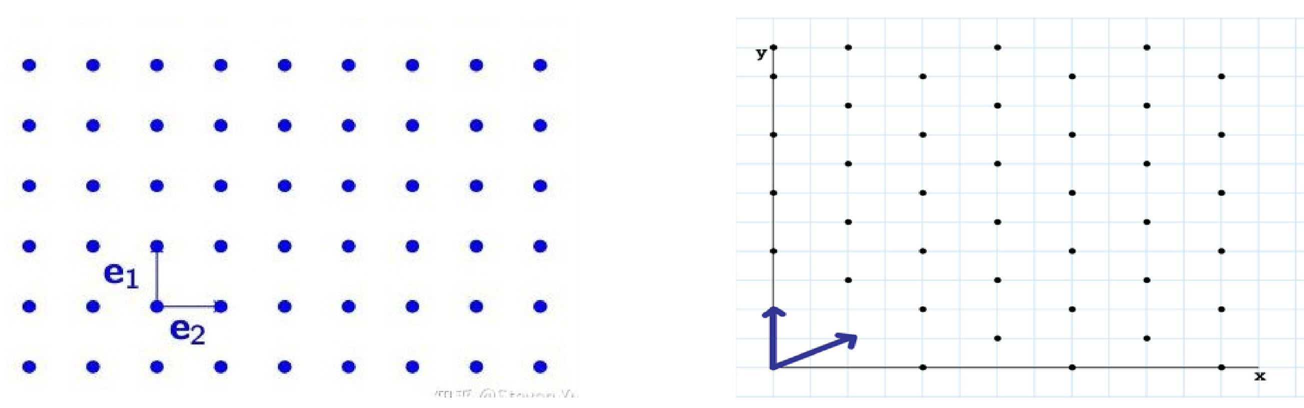


Figure: Covering of $\mathbb{R}^K$ with translations of the fundamental domain $\mathscr{F}$. $Covol(L) := Vol(\mathscr{F})$

For lattice packings,

$$\delta(L) = \frac{\Delta(L)}{Vol(B_1^n(0))} = \frac{\lambda(L)^n}{2^n \cdot covol(L)} \tag{1}$$

where $\delta(L)$ is called **center density**.



(a) The lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ (b) The hexagonal lattice in $\mathbb{R}^2$

Figure: Two different 2−dimensional lattices...

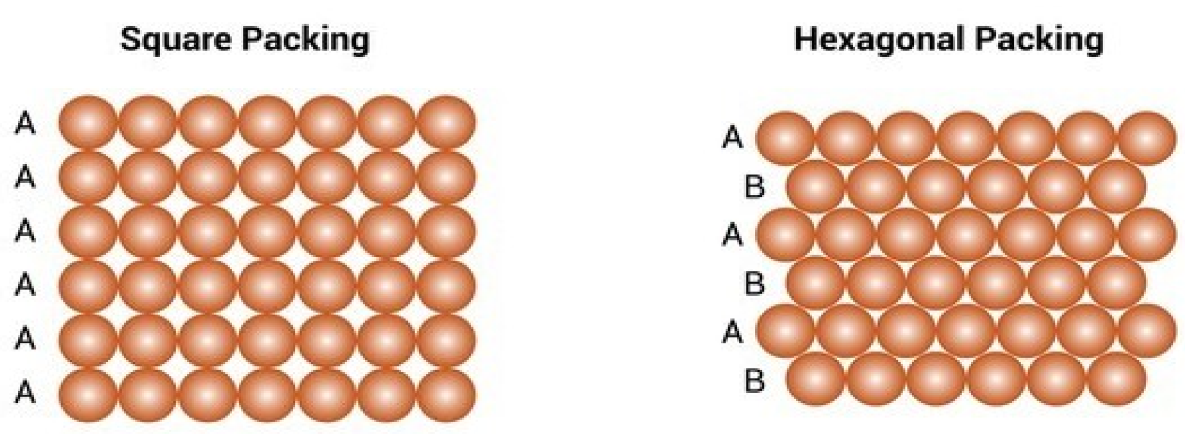


Figure: ...produce two different sphere packings! The hexagonal packing is optimal in $\mathbb{R}^2$.

Abstract Lattices

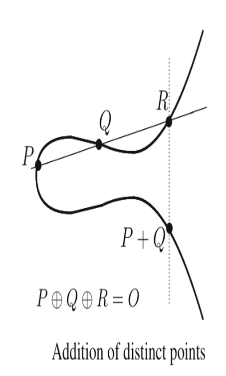
- More generally, a *lattice* $(L, \langle -,- \rangle_L)$ is a free abelian group of rank n with a $\mathbb{Z}-$bilinear positive-definite form.
- $\exists$ bijective map between

$$\{n-\text{dim. abstract lattices}\}/\cong \;\longleftrightarrow\; \{n-\text{dim. euclidean lattices}\}/O_n(\mathbb{R})$$

- **IDEA** (Shioda, Elkies): Construct dense sphere packings from Mordell-Weil lattices

Mordell-Weil Lattices - I

- Take an elliptic curve E over a global function field $K = k(t)$ with $k = \mathbb{F}_q$
- $E(K)$ (K-rational points of E) form an abelian group. **Mordell-Weil Theorem**: $E(K)$ is finitely generated (the **Mordell-Weil group** of E/K)
  - We can endow $E(K)$ with a positive-definite quadratic form $\hat{h}$
  - For $P = (x_P(t), y_P(t)) \in E(K)$, $h(P) := \deg(x_P(t))$. Then $\hat{h}(P) = \lim_{n\to+\infty} 4^{-n} h([2^n]P)$ is the *Néron-Tate height*
  - $L^E := (E(K)/E(K)_{\text{tors}}, \hat{h})$ is the **Mordell-Weil lattice of** E/K



Mordell-Weil Lattices - II

- For every place v of K, there is a "reduced curve" $\widetilde{E}_v$ over the (finite) residue field $\widetilde{K}_v$. For v of *bad reduction*, $\widetilde{E}_v$ is not smooth. A point of *bad reduction* gets reduced to the non-smooth point of $\widetilde{E}_v/\widetilde{K}_v$
- The *narrow Mordell-Weil group* $E_0(K)$ is the subgroup of points that have good reduction $\forall v$.
- The **L-function** of E/K is $L(E/K, T) := \prod_v L_v(E/K, T)^{-1}$ where the *local factor* $L_v(E/K, T)$ is a polynomial that encodes information about the $\#$ of points on $\widetilde{E}_v/\widetilde{K}_v$
- The **analytic rank** of E/K is $\rho(E/K) := \text{ord}_{T=|k|^{-1}} L(E/K, T)$

Computing the center density

Task: Given E/K, in order to find $\delta(L^E)$ (1) we need to compute:

1. The algebraic rank n (dimension of the lattice)
2. The shortest vector length $\lambda(L^E)$
3. The covolume of L^E, i.e. $Reg(E/K)^{1/2}$

$\hookrightarrow$ Theorem (Shioda):

$$\lambda(L_0^E) \geq \left(\frac{\deg(\Delta_{min}(E/K))}{6}\right)^{\frac{1}{2}}$$

$\hookrightarrow$ The **Birch-Swinnerton-Dyer Conjecture** (Millennium Problem!) is true for isotrivial elliptic curves over global function fields. Consequences:
(a) n (=algebraic rank of L^E) $= \rho(E/K)$ (analytic rank) !
(b) Explicit upper bound for $|Reg(E/K)|$ in terms of other arithmetic invariants of the curve
$\Longrightarrow$ lower bound for center density of the narrow Mordell-Weil lattice of E/K:

$$\delta(L_0^E) \geq \frac{\left(\frac{\deg(\Delta_{min}(E/K))}{24}\right)^{\frac{r}{2}}}{|E(K)_{tors}| \, |(L^*(E/K))|^{\frac{1}{2}} \, |k|^{\frac{g_C - 1}{2}} \, |H(E/K)|^{\frac{1}{2}} \, c(E/K)^{\frac{1}{2}}}$$

- Explicitly computing the L-function of E/K is the difficult part
- Other arithmetic invariants ($c(E/K), \Delta_{min}(E/K)$...) can be calculated using Tate's Algorithm

Notable results

- **Elkies** [1]: $y^2 + y = x^3 + t^{2^n+1} + \alpha_6$ over $K = \mathbb{F}_{2^{2n}}(t)$.
  We have $rk(L^{E_n}) = 2^{n+1}$. For $6 \leq n \leq 11$, $L_0^{E_n}$ provide the densest sphere packings in those dimensions (dim $= 128, \ldots, 4096$).
- **Leterrier** [2]: $y^2 + y = x^3 + bx + t^{3^n+1}$ over $K = \mathbb{F}_{3^{2n}}(t)$.
  We have $rk(L^{E_n}) = 2 \cdot 3^n$. For $n = 4, 5$, $L_0^{E_n}$ provide to this date the densest sphere packings in those dimensions (dim $= 162, 486$).

Laminated Lattices

An $(n+1)-$dim. *laminated* lattice packing is built by stacking layers of an $n-$dim. lattice packing so that spheres do not overlap (see image). There is a formal procedure to laminate abstract lattices as well. Now $\delta(\hat{L}_0^E)$ is a multiple of $\delta(L_0^E)$ with the factor depending on $\lambda(L_0^E)$ and the group L^E/L_0^E.

- If we have a dense $n-$dim. lattice packing, laminating it can be a good way to create a dense $(n+1)-$lattice packing
- To compute (a lower bound on) $\delta(\hat{L}_0^E)$, we need to find short vectors.

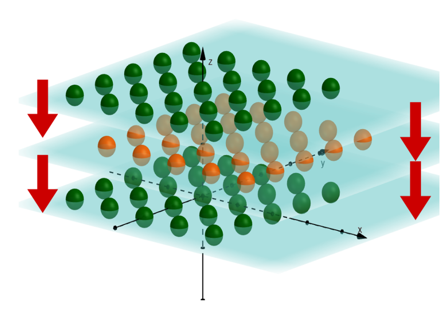


Figure: Laminated lattice packing: stacking layers

Laminated Lattices for $n+1 = 151, 307$ (Master's Thesis, 2025)

Laminating some of Leterrier's lattices ([2]) from $y^2 = x^3 + bx + t^{\frac{p+1}{2}}$ over $\mathbb{F}_{p^2}(t)$ for $p = 151$ (resp. $p = 307$ yields a lattice sphere packings of dim. p which has, to this date, highest center density in those dimensions, **provided** one can find a vector of length 26 (resp. 52), i.e. a polynomial of degree 26 (resp. 52) satisfying the curve equation. An open end...

References

[1] Noam D. Elkies.
Mordell-Weil lattices in characteristic 2. I. Construction and first properties.
Internat. Math. Res. Notices, (8):343 ff., approx. 18 pp. 1994.

[2] Gauthier Leterrier.
Some Mordell-Weil lattices and applications to sphere packings.
PhD thesis, EPFL, Lausanne, 2024.