Reduction of plane quartics and Dixmier-Ohno invariants

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Warming up: elliptic curves

Let E be an elliptic curve over $\mathbb{Q}$. Then E is given by a minimal Weierstraß equation

$$y^2 = x^3 + Ax + B$$

with coefficients in $\mathbb{Z}$. Let $p > 3$ be prime.

Lemma

Let $\Delta = -16(4A^3 + 27B^2)$ be the discriminant of E. Then

- *E has good reduction at p if $p \nmid \Delta$*
- *E has bad additive reduction if $p \mid A, \Delta$,*
- *E has bad multiplicative reduction if $p \mid \Delta$, but $p \nmid A$.*

After extending the base field, every curve will have either good reduction or bad multiplicative reduction, and we can distinguish the cases as follows.

Lemma

Let $j = 1728 \cdot \frac{-64A^3}{\Delta}$ be the j-invariant of E. Then E has potential good reduction if the denominator of j is not divisible by p, and potential bad multiplicative reduction otherwise.

Stable reduction and invariants

Definition

A **stable curve** over an algebraically closed field $\overline{K}$ is a proper, connected, 1-dimensional, reduced scheme C over K, such that

(i) all singular points are ordinary double points,

(ii) the arithmetic genus of C is at least 2,

(iii) every irreducible component of genus 0 has at least three singular points, counted with multiplicity.

Now consider the case of a curve C over a local field $\mathbb{Q}_p$ and suppose that the reduction of C modulo p is a stable curve.

Definition

The **stable reduction type** of C is the following combinatorial data:

- the number of irreducible components of $\mathcal{C}_s$ and their genera;
- for each singular point the two components it connects.

The goal of our work is to relate the stable reduction type of a curve to the invariants (e.g. j-invariants, Igusa-, Shioda-, Dixmier-Ohno invariants).

An example of stable reduction

After extending the local base field if necessary, any curve of genus at least 2 attains **stable reduction**. We will determine this reduction for a specific curve.

Example

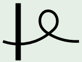
Suppose $p > 2$ and consider the genus 2 curve

$$H\colon y^2 = x(x-1)(x-2)(x-p)(x-p-1)(x-2p).$$

Reducing this modulo p we get a **genus 0 curve** with a cusp at $(x, y) = (0, 0)$ and a node at $(x, y) = (1, 0)$. If we "zoom in" on the cusp by substituting $x = px'$ and $y = p^{3/2}y'$ we get

$$y'^2 = x'(px'-1)(px'-2)(x'-1)(px'-p-1)(x'-2)$$

and the reduction is a **genus 1 curve** with a cusp at infinity.



It turns out the the **stable reduction** of H is the curve obtained by gluing these genus 0 and genus 1 curves at their cusps.

Situation in genus 2 (Liu, 1993)

Théorème 1. *Soit C une courbe projective lisse sur K, géométriquement connexe et de genre 2, soient $J_{2i}, 1 \leqq i \leqq 5$, les invariants de C associés à une équation $y^2 + Q(x)y = P(x)$. Alors on a*

(I) (Igusa) *$C_{\bar{s}}$ est lisse si et seulement si: $J_{2i}^5 J_{10}^{-i} \in R$ pour tout $i \leqq 5$;*

(II) *$\mathscr{C}_{\bar{s}}$ est irréductible avec un seul point double si et seulement si: $J_{2i}^6 I_{12}^{-i} \in R$ pour tout $i \leqq 5$ et $J_{10}^6 I_{12}^{-5} \in \mathfrak{m}$. La normalisation de $\mathscr{C}_{\bar{s}}$ est alors une courbe elliptique, d'invariant modulaire $j = \overline{(I_4^3 I_{12}^{-1})}$;*

(III) *$\mathscr{C}_{\bar{s}}$ est irréductible avec deux points doubles si et seulement si: $J_{2i}^2 I_4^{-i} \in R$ pour tout $i \leqq 5$, $J_{10}^2 I_4^{-5} \in \mathfrak{m}$, $I_{12} I_4^{-3} \in \mathfrak{m}$, et $J_4 I_4^{-1}$ ou $J_6^2 I_4^{-3}$ inversible dans R;*

(IV) *$\mathscr{C}_{\bar{s}}$ est constituée de deux droites projectives se coupant transversalement en trois points si et seulement si: $J_{2i}^2 I_4^{-i} \in \mathfrak{m}$ pour tout $2 \leqq i \leqq 5$;*

(V$_*$) *$\mathscr{C}_{\bar{s}}$ est la réunion de deux composantes irréductibles se coupant en un seul point si et seulement si:*

$$I_4^{\varepsilon} I_{2\varepsilon}^{-2} \in \mathfrak{m}, \quad J_{10}^{\varepsilon} I_{2\varepsilon}^{-5} \in \mathfrak{m}, \quad I_{12}^{\varepsilon} I_{2\varepsilon}^{-6} \in \mathfrak{m}, \tag{4}$$

où $I_{2\varepsilon}$ est définie dans (3) (cela implique que pour tout $i \leqq 5$, $J_{2i}^{\varepsilon} I_{2\varepsilon}^{-i} \in R$). De plus,

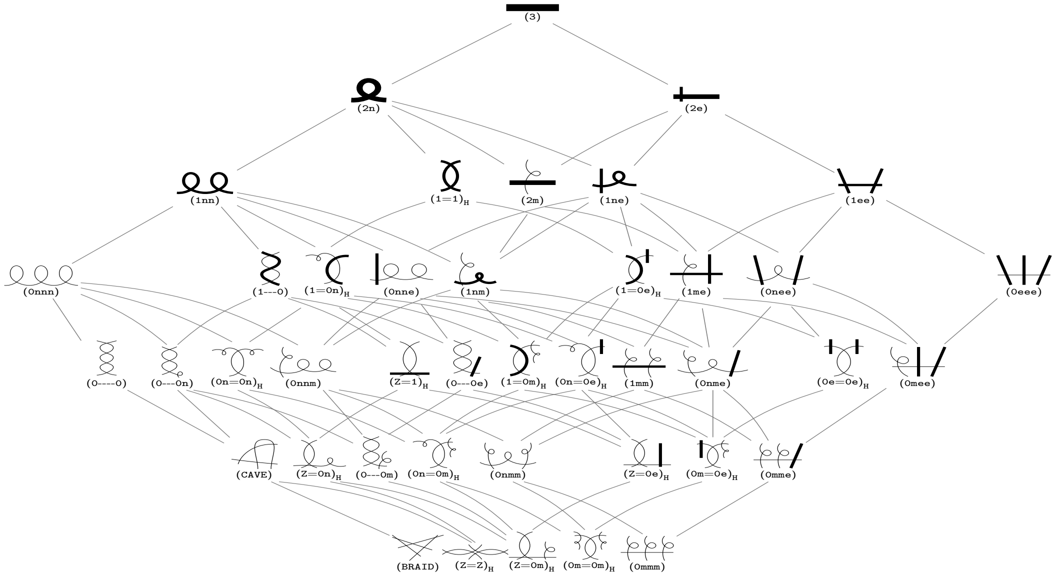
(V) *les composantes de $\mathscr{C}_{\bar{s}}$ sont lisses si et seulement si: en plus de (4), on a $I_4^{3\varepsilon} J_{10}^{-\varepsilon} I_{2\varepsilon}^{-1} \in R$, $I_{12}^{\varepsilon} J_{10}^{-\varepsilon} I_{2\varepsilon}^{-1} \in R$. Soient j_1, j_2 les invariants modulaires des deux composantes de $\mathscr{C}_{\bar{s}}$, alors*

$$\begin{cases} (j_1 j_2)^{\varepsilon} = \overline{(I_4^{3\varepsilon} J_{10}^{-\varepsilon} I_{2\varepsilon}^{-1})} \\ (j_1 + j_2)^{\varepsilon} = 2^6 \cdot 3^3 + \overline{(I_{12}^{\varepsilon} J_{10}^{-\varepsilon} I_{2\varepsilon}^{-1})}; \end{cases}$$

(VI) *une seule des deux composantes de $\mathscr{C}_{\bar{s}}$ est lisse si et seulement si: en plus de (4), on a $I_4^3 I_{12}^{-1} \in R$, $J_{10}^{\varepsilon} I_{2\varepsilon} I_{12}^{-\varepsilon} \in \mathfrak{m}$. L'invariant modulaire de la composante lisse de $\mathscr{C}_{\bar{s}}$ est alors $j = \overline{(I_4^3 I_{12}^{-1})}$;*

(VII) *les deux composantes de $\mathscr{C}_{\bar{s}}$ sont singulières si et seulement si: en plus de (4), on a $I_{12} I_4^{-3} \in \mathfrak{m}$, et $J_{10}^{\varepsilon} I_{2\varepsilon} I_4^{-3\varepsilon} \in \mathfrak{m}$.*

All 42 stable reduction types for genus 3 curves



Ternary quartics and Dixmier-Ohno invariants

A plane quartic curve is given by a ternary quartic form, i.e. a homogeneous degree 4 equation in 3 variables x, y, and z. A general ternary quartic is

$$\sum_{\substack{i,j,k=0,\dots,4 \\ i+j+k=4}} c_{ijk} x^i y^j z^k \in K[c_{004}, \dots, c_{400}][x,y,z].$$

The group SL_3 acts on these ternary quartics. We consider the ring of invariants

$$K[c_{004}, \dots, c_{400}]^{\mathrm{SL}_3}.$$

Dixmier (1987) and Ohno (2005) came up with a list of 13 invariants

$$I_3, I_6, I_9, J_9, I_{12}, J_{12}, I_{15}, J_{15}, I_{18}, J_{18}, I_{21}, J_{21}, I_{27}$$

that generate this ring, as long as $\mathrm{char}(K) > 7$.

Proposition

The map that associates to a smooth plane quartic curve, up to isomorphism, its Dixmier-Ohno invariants, considered as element of the weighted projective space $\mathbb{P}^{12}_{3,6,9,9,12,12,15,15,18,18,21,21,27}(K)$ is well defined and injective.

Singular quartics

In the PhD thesis of Hui, one can find a complete stratification of singular plane quartics in char. 0. Moreover, there is also a list of normal forms up to the action of SL_3. Here we use the Arnol'd classification to indicate the singularities, i.e. A_1 is a node, A_2 is a cusp, A_3 is a tacnode, et cetera.

Type	Dim.	Normal forms
Smooth 	6	$xz^3 + z(\alpha x^3 + \beta x^2 y + y^3) + \gamma x^4 + \delta x^3 y + \epsilon x^2 y^2 + \zeta xy^3 + y^4$
A_1 	5	$yz^3 + (\alpha y^2 + x^2)z^2 + (\beta y^3 + \gamma y^2 x + yx^2)z + \delta y^4 + \epsilon y^3 x$
A_2 	4	$yz^3 + (\alpha y^2 + \beta yx + x^2)z^2 + (\gamma y^3 + \delta y^2 x)z + y^3 x$
A_3 	3	$x^2 z^2 + \alpha y^2 xz + y^4 + \beta y^3 x + \gamma y^2 x^2 + yx^3$
$\vdots$ $\vdots$	$\vdots$	$\vdots$

There can be multiple singularities ($A_1^2 A_2$), reducible quartics, and double components (conic^2).

Definition

If at least one of the Dixmier-Ohno invariants of such a normal form is non-zero, then the singularity type is called **GIT-semi-stable**.

Relating Dixmier-Ohno invariants with singularity type

First we find algebraic relations in the Dixmier-Ohno invariants that are satisfied by all quartics with a certain singularity type.

Algorithm

Step 1. Given a normal form with parameters $\alpha, \ldots,$, generate a lot of random examples by picking random values for $\alpha, \ldots$.

Step 2. For each of these examples, compute the Dixmier-Ohno invariants.

Step 3. For some small values of d, use linear algebra to determine if there are any homogeneous relations of degree d in the Dixmier-Ohno variants that are satisfied by all the tuples found in **Step 2**.

Step 4. Evaluate the relation in the Dixmier-Ohno invariants of the normal form to verify these relations hold for all forms.

Note that for some of the singularity types, it is impossible to distinguish between them using just Dixmier-Ohno invariants.

Example

The quartics $x^2z^2 + y^4 + yx^3$ and $x^2z^2 + y^4 + y^3z + y^2z^2$ have the same Dixmier-Ohno invariants, even though the first quartic has singularity type A_3 and the second has singularity type A_1A_3.

Relating Dixmier-Ohno invariants with singularity type

We still need to prove that we found enough equations to determine the singularity type. The following example illustrates a strategy to prove this.

Example

Consider the normal form

$$(y^2 + \alpha yx + x^2)z^2 + (\beta y^2 x + \gamma yx^2)z + y^2 z^2$$

for a quartic of singularity type A_1^3. Suppose that the Dixmier-Ohno invariants of this quartic also satisfy the relations for the type ${}^{r}A^4_{1,\mathrm{con}}$. Expressing these relations in terms of the parameters α, β, γ, by doing a Gröbner basis computation, we obtain

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4 = 0.$$

We can then check that such quartics have 4 singular points, namely

$$(1:0:0), (0:1:0), (0:0:1), \left(\frac{2\beta - \alpha\gamma}{\gamma^2 - 4} : \frac{2\gamma - \alpha\beta}{\beta^2 - 4} : 1\right).$$

Hence, such quartics are not of singularity type A_1^3.

An algorithm to determine the singularity type

Algorithm

Input : Dixmier-Ohno invariants $(I_3 : I_6 : \cdots : I_{27}) \in \mathbb{P}^{12}_{3,6,\ldots,27}(\overline{K})$ of a plane quartic, $\mathrm{char}(\overline{K}) = 0$
Output: A set of possible singularity types for the plane quartic

`// Easy cases`
If $I_{27} \neq 0$, then **return** { Smooth }
If $(I_3 : I_6 : \cdots : I_{27}) = 0$, then **return** { Unstable }

`// Dimension 0`
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V({}^r\mathrm{A}_1^6)$, then **return** $\{ {}^r\mathrm{A}_1^6 \}$
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathrm{A}_2^3)$, then **return** $\{ \mathrm{A}_2^3 \}$
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathrm{A}_4)$, then **return** $\{ \mathrm{A}_4, \mathrm{A}_5, \mathrm{A}_6, \mathrm{A}_1\,\mathrm{A}_4, \mathrm{A}_2\,\mathrm{A}_4, {}^r\mathrm{A}_7, {}^r\mathrm{A}_1\,\mathrm{A}_5, c^2 \}$
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V({}^r\mathrm{A}_1\mathrm{A}_3)$, then **return** $\{ {}^r\mathrm{A}_1\,\mathrm{A}_3, {}^r\mathrm{A}_1^2\,\mathrm{A}_3, {}^r\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3, {}^r\mathrm{A}_1\,\mathrm{A}_3^2, {}^r\mathrm{A}_1^3\,\mathrm{A}_3 \}$

`// Dimension 1`
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V({}^r\mathrm{A}_1^5)$, then **return** $\{ {}^r\mathrm{A}_1^5 \}$
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V({}^r\mathrm{A}_1^3\mathrm{A}_2)$, then **return** $\{ {}^r\mathrm{A}_1^3\,\mathrm{A}_2 \}$
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathrm{A}_1\mathrm{A}_2^2)$, then **return** $\{ \mathrm{A}_1\,\mathrm{A}_2^2 \}$
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathrm{A}_3)$, then **return** $\{ \mathrm{A}_3, \mathrm{A}_1\,\mathrm{A}_3, \mathrm{A}_2\,\mathrm{A}_3, {}^r\mathrm{A}_3^2, {}^r\mathrm{A}_1^2\,\mathrm{A}_3 \}$

`// Dimension 2, 3, etc.`
...

Relating stable reduction type with singularity type: cusps

Suppose $C/\mathbb{Q}_p$ is a plane quartic and the reduction $C_p/\mathbb{F}_p$ is GIT-semi-stable. Then the singularity type of C_p gives some information about the stable reduction type of C. Wherever there are nodes on C_p, there will also be nodes in the stable reduction of C. Even in the case of cusps and worse singularities we can get some information.

Theorem

If you "zoom in" on a cusp on C_p, you will see a curve of arithmetic genus 1. In fact, the (singular) plane quartic that you get will factor as $(cubic) \cdot (line)$ and the line and the cubic intersect in an A_5*-singularity.*

It is impossible to determine from C_p only whether the arithmetic genus 1 curve that you get is a smooth curve of genus 1 or a curve of genus 0 with one self-intersection.

Remark

"Zooming in" on a cusp, means putting C_p in a standard form and then doing a change of coordinates to see the rest of the curve, as in T. Dokchitser's Δ_v-regular models (T. Dokchitser, 2021).

Relating stable reduction type with singularity type: tacnodes

Tacnodes hide two genus 1 components intersecting in two points.

Theorem

If C_p has a tacnode (A_3-singularity), then the rest of C_p is of arithmetic genus 1, and upon "zooming in" on the tacnode, one finds another curve of arithmetic genus 1, possibly a genus 0 with a node or cusp, with a tacnode.
The two arithmetic genus 1 parts intersect in two points.

Finally, A_4-singularities and the case of the square of a conic give hyperelliptic reduction.

Theorem

If C_p has a A_4-singularity or is $(conic)^2$, we can find a so-called toggle model for C of the shape

$$Q^2 + \pi^s G = 0,$$

where Q and G are homogeneous of degree 2 and 4 with integral coefficients such that $Q \mod p$ is irreducible and $G \mod p$ is non-zero.
By the theory of toggle models, the reduction is a hyperelliptic curve.

From singularity type to stable reduction types

Sing. type	Red. type
A_1	(2n)
A_1^2	(1nn)
A_1^3	(0nnn)
${}^rA_1^3$	(1---0)
${}^rA_1^4$ *con*	(0----0)
${}^rA_1^4$ *cub*	(0---0n)
${}^rA_1^5$	(CAVE)
${}^rA_1^6$	(BRAID)

(a) A_1-singularities

Sing. type	Red. type
Smooth	(3)
$\geq A_4$, incl. c^2	$(*)_{\mathrm{H}}$

(b) Other types

Sing. type	Red. type
A_2	(2e) or (2m)
A_1A_2	(1ne) or (1nm)
$A_1^2A_2$	(0nne) or (0nnm)
${}^rA_1^3A_2$	(0---0e) or (0---0m)
A_2^2	(1ee) or (1em) or (1mm)
$A_1A_2^2$	(0nee) or (0nem) or (0nmm)
A_2^3	(0eee) / (0eem) / (0emm) / (0mmm)

(c) A_2-singularities

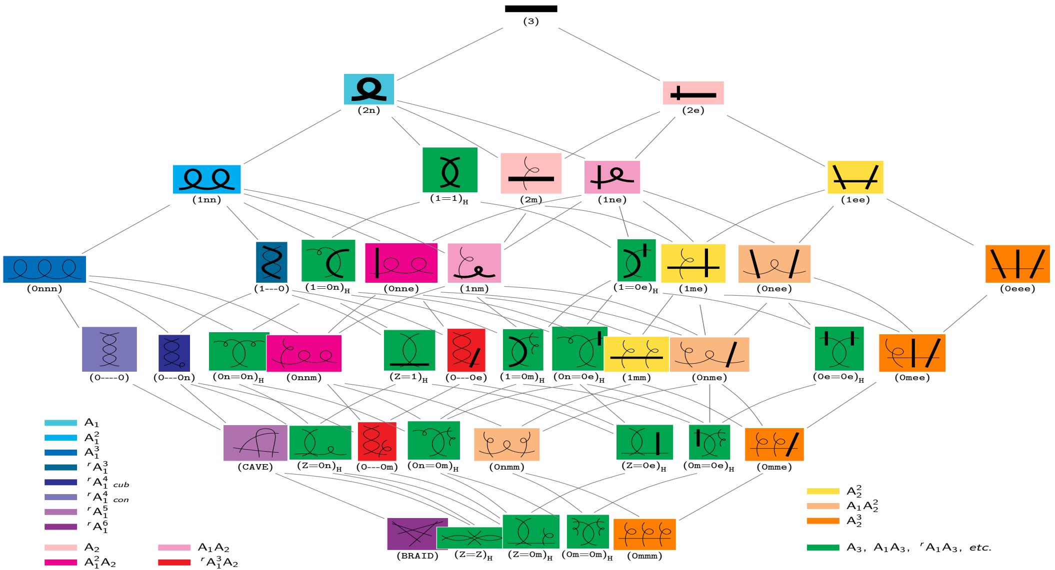
Sing. type	Red. type
A_3	$(1 = *)_{\mathrm{H}}$
A_1A_3	$(0n = *)_{\mathrm{H}}$ or $(* = 0n)_{\mathrm{H}}$
${}^rA_1^2A_3$ *con*	$(Z = *)_{\mathrm{H}}$
A_2A_3	$(* = 0e)_{\mathrm{H}}$ or $(* = 0m)_{\mathrm{H}}$
${}^rA_3^2$	$(1 = *)_{\mathrm{H}}$ or $(0n = *)_{\mathrm{H}}$ or $(Z = *)_{\mathrm{H}}$

(d) A_3-singularities

Sing. type	Red. type
${}^rA_1A_3$	$(1 = *)_{\mathrm{H}}$ or $(* = 1)_{\mathrm{H}}$
${}^rA_1^2A_3$ *cub*	$(0n = *)_{\mathrm{H}}$ or $(* = 0n)_{\mathrm{H}}$
${}^rA_1^3A_3$	$(Z = *)_{\mathrm{H}}$
${}^rA_1A_2A_3$	$(* = 0e)_{\mathrm{H}}$ or $(* = 0m)_{\mathrm{H}}$
${}^rA_1A_3^2$	$(1 = *)_{\mathrm{H}}$ or $(0n = *)_{\mathrm{H}}$ or $(Z = *)_{\mathrm{H}}$

(e) ${}^rA_3$-singularities

From stable reduction type to singularity type



An algorithm to (almost) determine the stable reduction type

Algorithm

Input: Dixmier-Ohno invariants $(I_3 : I_6 : \cdots : I_{27}) \in \mathbb{P}_{3,6,\ldots,27}(\overline{\mathbb{Q}_p})$ of a plane quartic C.
Output: A list of possible stable reduction types for the plane quartic.

Step 1. Rescale the invariants so that they are all integers, but not all of them reduce to 0 mod p.
Step 2. Run the previous algorithm to determine the singularity type of C_p.
Step 3. Use the previous table to determine the possible stable reduction types for C.

The computations that we did only verified that this algorithm is correct when $7 < p < 100$.

Conjecture

The algorithm is correct for any prime $p > 7$.

For each characteristic p, we made a program that you could run to actually verify the correctness of the algorithm. This program runs in a reasonable time.

Question

Why use this algorithm when there are already other algorithms out that can completely determine the stable reduction type (e.g. resolution of singularities)?

Computation on a database

Answer

Because it is wicked fast!

We ran the algorithm on a dataset of more than 82 000 plane quartics, by Sutherland, with 137 496 pairs of (C, p) where $p > 7$ is a prime of bad reduction.

For the vast majority of the curves we could uniquely determine the stable reduction type and on average this took about 10 milliseconds per pair (C, p).

Singularity type	Stable reduction type	Number of times
A_1	(2n)	131 673
A_1^2	(1nn)	3511
A_2	(2e) or (2n)	1829

Table: Most common reduction types

Summary

Plane quartics

↓

Dixmier-Ohno invariants

↓

Singularity type

↓

Stable reduction type