Sesquilinear pairings on elliptic curves (+ isogenies)

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Bilinear pairings

Let A, B, R be abelian groups. Let

$$\langle \cdot, \cdot \rangle : A \times B \to R$$

be linear in each factor.

Our interest: A and B groups of points on an elliptic curve.

Weil and Tate pairings

Weil pairing:

$$e_m : E(K)[m] \times E(K)[m] \to \mu_m$$

Tate pairing:

$$t_m : E(K)[m] \times E(K)/mE(K) \to K^*/(K^*)^m$$

Implies fun cryptography. Example:

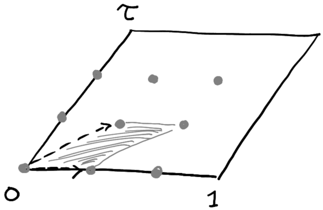
$$t_m([a]P, [b]Q)^c = t_m(P, Q)^{abc}.$$

Weil pairing over $\mathbb{C}$

Weil pairing over $\mathbb{C}$ (Galbraith has nice notes):

Let 1 and τ form a basis for Λ giving $E \cong \mathbb{C}/\Lambda$:

$$e_m\left(\frac{a+b\tau}{m}, \frac{c+d\tau}{m}\right) = e^{2\pi i \frac{ad-bc}{m}}.$$



Paths for homology of torus: $\gamma_1 : 0 \to 1$ and $\gamma_\tau : 0 \to \tau$.

$$(a\gamma_1 + b\gamma_\tau)\cdot(c\gamma_1 + d\gamma_\tau) = ad - bc.$$

Extensions

An extension

$$0 \longrightarrow \mathbb{G}_m \longrightarrow X \longrightarrow E \longrightarrow 0$$

is given by a factor set

$$f : E \times E \to \mathbb{G}_m$$

determining the group law on X via

$$(x,P)(y,Q) = (xyf(P,Q), P+Q).$$

Extensions

An extension

$$0 \longrightarrow K^* \longrightarrow X \longrightarrow E(K) \longrightarrow 0$$

is given by a factor set

$$f : E(K) \times E(K) \to K^*$$

determining the group law on X via

$$(x, P)(y, Q) = (xy\, f(P, Q), P + Q).$$

Monodromy

A group fact in E,

$$\sum P_i = \mathscr{O},$$

$$\Downarrow$$

a monodromy $\alpha \in K^*$:

$$\sum (x_i, P_i) = \left(\left(\prod x_i\right)\alpha, \mathscr{O}\right).$$

A biextension X ‘glues together’ many extensions:

X has action of K^* with quotient

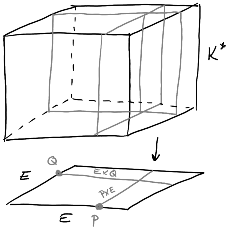
$$\pi : X \to E \times E$$

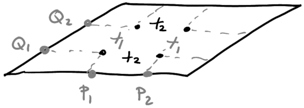
where fibres $X_{(P,Q)}$ are homogeneous spaces for K^*.

There are two compatible operations:

1. $+_1$ defined on $X_{\{P\}\times E}$;
2. $+_2$ defined on $X_{E\times\{Q\}}$.

Each $X_{\{P\}\times E}$
is an extension of E by K^* determined by P, and similarly.





The Poincaré biextension

In our case X is given by a biextension factor set

$$f : E \times E \times E \to K^*$$

so that f restricts to a factor set on $E \times E \times \{Q\}$ and $\{P\} \times E \times E$.

Let f be the rational function with divisor

$$C := m_{123}^*(\mathscr{O}) - m_{12}^*(\mathscr{O}) - m_{23}^*(\mathscr{O}) - m_{13}^*(\mathscr{O}) + m_1^*(\mathscr{O}) + m_2^*(\mathscr{O}) + m_3^*(\mathscr{O}).$$

Has an expression in terms of elliptic nets:

$$\frac{W(P+Q+R)W(P)W(Q)W(R)}{W(P+Q)W(Q+R)W(P+R)}.$$

Monodromy

Fixing Q in E, we have an extension $X_{E\times\{Q\}}$.

If $P \in E[m]$, then the group fact $mP = \mathcal{O}$ gives a monodromy on $X_{E\times\{Q\}}$.

This is the Tate pairing $t_m(P, Q)$.

The Weil pairing is the quotient

$$e_m(P, Q) = \frac{\text{monodromy of } mP = \mathcal{O} \text{ on } X_{E\times\{Q\}}}{\text{monodromy of } mQ = \mathcal{O} \text{ on } X_{\{P\}\times E}}$$

Weil and Tate pairings from monodromy

The extension $X_{E\times\{Q\}}$ has factor set

$$E\times E\to K^*,\quad (P,R)\mapsto f_{P,R}((Q)-(\mathscr{O}))$$

where

$$\mathrm{div}(f_{P,R})=(P+R)-(P)-(R)+(\mathscr{O}).$$

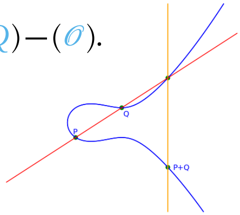
Gives rise to Tate pairing formula:

$$t_m : E(K)[m]\times E(K)/mE(K)\to K^*/(K^*)^m$$

$$t_m(P,Q)=f_P(D_Q),\quad \mathrm{div}(f_P)=m(P)-m(\mathscr{O}),\quad D_Q\sim(Q)-(\mathscr{O}).$$

Tate pairing computation (Miller's Algorithm)

$$t_m(P, Q) = f_P(D_Q), \quad \mathrm{div}(f_P) = m(P) - m(\mathcal{O}), \quad D_Q \sim (Q) - (\mathcal{O}).$$



1. Create double-and-add chain of operations $k_1 + k_2$ for m.
2. This
gives a double-and-add chain of divisors $D_k := k(P) - ([k]P) - (k-1)(\mathcal{O})$
satisfying $D_{k_1} + D_{k_2} \sim D_{k_1+k_2}$. Note that $\mathrm{div}(f_P) = D_m$.
3. Each step
$D_{k_1+k_2} - D_{k_1} - D_{k_2} = ([k_1]P) + ([k_2]P) - ([k_1+k_2]P) - (\mathcal{O})$ is an instance
of the group law, i.e. a rational function f_{k_1,k_2}. Thus $f_P = \prod f_{k_{i,1},k_{i,2}}$.
4. Compute the double-and-add chain to
compute $f_P(D_Q) = \prod f_{k_{i,1},k_{i,2}}(D_Q)$ (always evaluated, i.e. elements of K^*).

Sesquilinear pairings

Let $\alpha, \beta \in \mathcal{O}$, an order in an imaginary quadratic field. A sesquilinear pairing is a bilinear pairing with:

$$\langle \alpha P, \beta Q \rangle = \langle P, Q \rangle^{\alpha \overline{\beta}}.$$

(We can also do everything today with $\mathcal{O}$ a quaternion order, at the cost of lots of extra notation.)

Calculus of $\mathscr{O}$-divisors

Extend scalars:

$$\mathrm{Div}_{\mathscr{O}}(E) := \mathscr{O} \otimes_{\mathbb{Z}} \mathrm{Div}(E).$$

We also extend scalars on $K(E)^*$ and K^*, writing multiplicatively, e.g. g^{1+i}.
Principal divisors:

$$\mathrm{div}\left(\prod_i f_i^{\tau_i}\right) = \sum_i \tau_i \,\mathrm{div}(f_i).$$

Then

$$\mathrm{Pic}^0_{\mathscr{O}}(E) := \mathscr{O} \otimes_{\mathbb{Z}} \mathrm{Pic}^0(E).$$

Evaluating an $\mathcal{O}$-function at an $\mathcal{O}$-divisor

If f and D are usual function and divisor, then

$$f^{\alpha}(\beta \cdot D) := f(D)^{\alpha\overline{\beta}}.$$

This gives $\mathcal{O}$-Weil reciprocity:

$$f(\mathrm{div}(g)) = \overline{g(\mathrm{div}(f))},$$

where conjugation acts on the scalars.

Weil and Tate pairings

Recall: $E \cong \mathrm{Pic}^0(E)$, $\quad P \mapsto (P) - (\mathcal{O})$.

$$e_m : \mathrm{Pic}^0(E)[m] \times \mathrm{Pic}^0(E)[m] \to \mathbb{G}_m[m],$$
$$t_m : \mathrm{Pic}^0(E)[m] \times \mathrm{Pic}^0(E)/[m]\mathrm{Pic}^0(E) \to \mathbb{G}_m/(\mathbb{G}_m)^m,$$

given by

$$t_m(D_P, D_Q) = f_P(D_Q) \quad \text{where} \quad \mathrm{div}(f_P) \sim m \cdot D_P,$$
$$e_m(D_P, D_Q) = \frac{f_P(D_Q)}{f_Q(D_P)}.$$

Galois invariant, sesquilinear, compatible, etc.

Sesquilinear pairings

$$
\begin{aligned}
W_\alpha &: \mathrm{Pic}^0_{\mathcal{O}}(E)[\overline{\alpha}] \times \mathrm{Pic}^0_{\mathcal{O}}(E)[\alpha] \to \mathbb{G}_m^{\otimes_{\mathbb{Z}} \mathcal{O}}[\overline{\alpha}],\\
T_\alpha &: \mathrm{Pic}^0_{\mathcal{O}}(E)[\overline{\alpha}] \times \mathrm{Pic}^0_{\mathcal{O}}(E)/[\alpha]\mathrm{Pic}^0_{\mathcal{O}}(E) \to \mathbb{G}_m^{\otimes_{\mathbb{Z}} \mathcal{O}}/(\mathbb{G}_m^{\otimes_{\mathbb{Z}} \mathcal{O}})^{\overline{\alpha}},
\end{aligned}
$$

given by

$$
T_\alpha(D_P, D_Q) = f_P(D_Q) \quad \text{where} \quad \mathrm{div}(f_P) \sim \overline{\alpha} \cdot D_P,
$$

$$
W_\alpha(D_P, D_Q) = \frac{f_P(D_Q)}{f_Q(D_P)}.
$$

Galois invariant, sesquilinear, compatible, etc.

Moving from formal to CM by $\mathscr{O} = \mathbb{Z}[\tau]$

$$0 \longrightarrow E \xrightarrow{\eta} \mathrm{Pic}^0_{\mathscr{O}}(E) \xrightarrow{\epsilon} E \longrightarrow 0$$

$$\epsilon : \quad D_1 + \tau \cdot D_2 \quad \mapsto \quad D_1^{\Sigma} + [\tau] D_2^{\Sigma}.$$

$$\eta : \quad P \quad \mapsto \quad ([-\tau]P) - (\mathscr{O}) + \tau((P) - (\mathscr{O})).$$

where $\left(\sum \alpha_i (P_i)\right)^{\Sigma} = \sum_i [\alpha_i] P_i$.

A Weil-like pairing

$$\widehat{W}_\alpha : E[\overline{\alpha}] \times E[\alpha] \to \mathbb{G}_m^{\otimes_{\mathbb{Z}} \mathscr{O}}[\alpha].$$

1. Well-defined, bilinear, Galois invariant, **non-degenerate**.
2. Sesquilinearity:
$$\widehat{W}_\alpha([\gamma]P, [\delta]Q) = \widehat{W}_\alpha(P,Q)^{\delta\overline{\gamma}}.$$
3. Conjugate skew-Hermitian:
$$\widehat{W}_\alpha(P,Q) = \overline{\widehat{W}_{\overline{\alpha}}(Q,P)}^{\,-1}.$$
4. Compatibility: Let $\phi : E \to E'$ respect CM by $\mathscr{O}$.
$$\widehat{W}_\alpha(\phi P, \phi Q) = \widehat{W}_\alpha(P,Q)^{\deg \phi}.$$
5. Coherence:
$$\widehat{W}_{\alpha\beta}(P,Q) = \widehat{W}_\alpha([\overline{\beta}]P, Q), \quad \widehat{W}_{\alpha\beta}(P,Q) = \widehat{W}_\beta(P, [\alpha]Q).$$

A Tate-like pairing

$$\widehat{T}_\alpha : E[\overline{\alpha}] \times E/[\alpha]E \to \mathbb{G}_m^{\otimes_{\mathbb{Z}} \mathcal{O}} / (\mathbb{G}_m^{\otimes_{\mathbb{Z}} \mathcal{O}})^\alpha.$$

1. Well-defined, bilinear, Galois invariant, **non-degenerate**.
2. Sesquilinearity:
$$\widehat{T}_\alpha([\gamma]P, [\delta]Q) = \widehat{T}_\alpha(P, Q)^{\overline{\gamma}\delta}.$$
3. Compatibility: Let $\phi : E \to E'$ respect CM by $\mathcal{O}$.
$$\widehat{T}_\alpha(\phi P, \phi Q) = \widehat{T}_\alpha(P, Q)^{\deg \phi}.$$
4. Coherence:
$$\widehat{T}_{\alpha\beta}(P, Q) \bmod (\mathbb{G}_m^{\otimes_{\mathbb{Z}} R})^\alpha = \widehat{T}_\alpha([\overline{\beta}]P, Q \bmod [\alpha]E).$$
$$\widehat{T}_{\alpha\beta}(P, Q) \bmod (\mathbb{G}_m^{\otimes_{\mathbb{Z}} R})^\beta = \widehat{T}_\beta(P, [\alpha]Q \bmod [\beta]E).$$

In terms of usual Weil and Tate pairings

Let $\mathcal{O} = \mathbb{Z}[\tau]$.

$$\widehat{T}_n(P,Q) = \left(t_n(P,Q)^{2N(\tau)} t_n([-\overline{\tau}]P,Q)^{Tr(\tau)}\right)(t_n([\overline{\tau}-\tau]P,Q))^{\tau}.$$

Furthermore, provided both of the following quantities are defined,

$$\widehat{T}_{N(\alpha)}(P,Q) = \widehat{T}_{\alpha}(P,Q)^{\overline{\alpha}} \pmod{(\mathbb{G}_m^{\otimes_{\mathbb{Z}} R})^{\alpha}}$$

Remark: Let $\langle x, y \rangle$ be a bilinear pairing on $\mathbb{Z}[\tau]$. Then

$$\langle x_1 + \tau x_2, y_1 + \tau y_2 \rangle := \langle x_1, y_1 \rangle + N(\tau)\langle x_2, y_2 \rangle + Tr(\tau)\langle x_1, y_2 \rangle + \tau(\langle x_2, y_1 \rangle - \langle x_1, y_2 \rangle)$$

defines a sesquilinear pairing.

Generalized pairings: Bruin, Garefalakis, Robert, Castryck-Houben-Merz-Mula-van Buuren-Vercauteren

Computation of $\widehat{T}_{\alpha}(P, Q)$

Suppose

$$\overline{\alpha} = d - c\tau, \quad \overline{\alpha}\tau = -b + a\tau.$$

For $P \in E[\overline{\alpha}]$, $f_P = f_{P,1} f_{P,2}^{\tau}$ with

$$\operatorname{div}(f_{P,1}) = a([-\tau]P) + b(P) - (a+b)(\mathcal{O}), \quad \operatorname{div}(f_{P,2}) = c([-\tau]P) + d(P) - (c+d)(\mathcal{O}).$$

Auxiliary point S; take $D_Q = D_{Q,1} + \tau \cdot D_{Q,2}$ with

$$D_{Q,1} = ([-\tau]Q + [-\tau]S) - ([-\tau]S), \quad D_{Q,2} = (Q+S) - (S).$$

Then

$$\widehat{T}_{\alpha}(P, Q) := f_P(D_Q) =$$

$$\left(f_{P,1}(D_{Q,1}) f_{P,1}(D_{Q,2})^{Tr(\tau)} f_{P,2}(D_{Q,2})^{N(\tau)}\right)\left(f_{P,2}(D_{Q,1}) f_{P,1}(D_{Q,2})^{-1}\right)^{\tau}.$$

Applications to Isogenies (joint with Joseph Macula)

Finite $\mathbb{F}$. There is a faithful action of $\mathrm{Cl}(\mathcal{O})$ on

$$\mathrm{Ell}(\mathcal{O}) = \{E/\mathbb{F} : E \text{ has CM by } \mathcal{O}\}.$$

When $\mathfrak{a} \cdot E_1 = E_2$, this gives an isogeny $\phi_{\mathfrak{a}} : E_1 \to E_2$ respecting $\mathcal{O}$.

Hard Problem 1: Given E_1 and $E_2 \in \mathrm{Ell}(\mathcal{O})$, find $\phi : E_1 \to E_2$ respecting $\mathcal{O}$.

Hard Problem 2: Given E_1 and $E_2 \in \mathrm{Ell}(\mathcal{O})$, and $\deg \phi$, find $\phi : E_1 \to E_2$ respecting $\mathcal{O}$.

Isogeny interpolation
(Castryck-Decru-Maino-Martindale-Panny-Pope-Robert-Wesolowski)

Hard Problem 2: Given E_1 and $E_2 \in \mathrm{Ell}(\mathcal{O})$, and $\deg \phi$, find $\phi : E_1 \to E_2$ respecting $\mathcal{O}$.

Wouter's talk [CDM+24]: to efficiently determine $\phi : E_1 \to E_2$, it suffices to find $\phi(G)$ (actually ϕ of generators) for some subgroup G of size at least $4 \deg \phi + 1$.

Hard Problem 3: Given E_1 and $E_2 \in \mathrm{Ell}(\mathcal{O})$, and $\deg \phi$, find $\phi(G)$, $\#G > 4 \deg \phi$ for $\phi : E_1 \to E_2$ respecting $\mathcal{O}$.

Recovering an isogeny

An idea of Castryck-Houben-Merz-Mula-van Buuren-Vercauteren: Let $m > 4\deg\phi$.
Suppose $P \in E_1[m]$, and suppose $\phi(P) = kP' \subseteq E_2[m]$. Use a pairing:

$$\langle P, P\rangle^{\deg\phi} = \langle \phi P, \phi P\rangle = \langle kP', kP'\rangle = \langle P', P'\rangle^{k^2}.$$

So

$$P, P', \deg\phi \xrightarrow{\text{discrete log}} k^2 \pmod m \implies \phi P = kP' \xrightarrow{\text{isog. interp.}} \phi.$$

Challenges:

1. Make sure $\phi P \in \mathbb{Z}P'$.
2. Make sure $\langle P, P\rangle$ is non-trivial.

CHMMvBV: Non-degenerate self-pairings when $m \mid \Delta_{\mathcal{O}}$.

$\mathcal{O}$-sesquilinear pairings

Let $m^2 > 4\deg\phi$. Suppose $P \in E_1[m]$, suppose $\mathcal{O}P' = E_2[m]$. Use a pairing:

$$\langle P, P\rangle^{\deg\phi} = \langle \phi P, \phi P\rangle = \langle \lambda P', \lambda P'\rangle = \langle P', P'\rangle^{N(\lambda)}.$$

So

$$P, P', \deg\phi \quad \xRightarrow{\text{discrete log}} \quad N(\lambda) \pmod m \quad \not\Rightarrow \quad \phi P = [\lambda]P' \quad \xRightarrow{\text{isog. interp.}} \quad \phi.$$

Pros/Cons:

1. Easier to guarantee $\mathcal{O}P' = E_2[m]$.
2. Easier to obtain non-degenerate pairings (m coprime to $\Delta_{\mathcal{O}}$).
3. For m coprimes to $\Delta_{\mathcal{O}}$, knowing $N(\lambda) \pmod m$ only cuts down $\lambda \pmod m$ from $\sim m^2$ options to $\sim m$ options.
4. For $m \mid \Delta_{\mathcal{O}}$ it works! And is sometimes more efficient.

When is $\widehat{T}_m$ non-degenerate?

Let $\eta : \mathcal{O} \to \mathrm{End}(E)$ extend to $\eta : K \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}(E)$.

Proposition (Macula-S.)

$E[m]$ is $\mathcal{O}$-cyclic as an $\mathcal{O}$-module if and only if m is coprime to $[\eta(K) \cap \mathrm{End}(E) : \eta(\mathcal{O})]$.

Based on results of Lenstra.

Theorem (Macula-S.)

Suppose $\mu_m \subseteq \mathbb{F}$, m is coprime to $\Delta_{\mathcal{O}}$, and $E[m] \subseteq E(\mathbb{F})$. Then $\widehat{T}_n(P,P)$ has full order whenever $\mathcal{O}P = E[m]$.

Computation of $\mathcal{O}$-pairings

Theorem (Macula-S.)

Suppose computations in $\mathbb{F}$ *and* $E[m]$*, and discrete logarithms in* μ_m *are all efficient.*
Let m *be coprime to* $\Delta_{\mathcal{O}}$ *and let* $\mathcal{O} \subseteq \mathrm{End}(E)$.
Then

$$\textit{computation of the } \mathcal{O}\textit{-pairing } \widehat{T}_m(P,Q) \textit{ on } E[m]$$

$$\textit{is equivalent to}$$

$$\textit{computation of } \mathcal{O} \textit{ acting on } E[m].$$

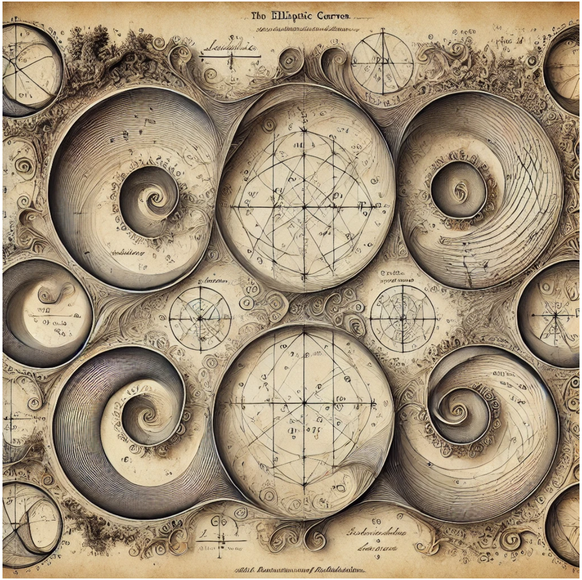
A trade-off (Supersingular case)

Suppose $m^2 > 4 \deg \phi$, coprime to $\Delta_{\mathscr{O}}$ and $\deg \phi$.

Situations where we can obtain ϕ:

Know full $\phi(E_1[m])$	Known $\phi(\langle P \rangle)$, $P \in E_1[m]$ and $\mathscr{O} \subseteq \mathrm{End}(E_i)$ (S.-Macula)	Know full $\mathrm{End}(E_i)$

Thank you!



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