A Heuristic Subexponential Algorithm to Find Paths in Markoff Graphs over Finite Fields

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Algorithmic Number Theory Symposium
(ANTS XVI), MIT

Friday July 19, 2024, 12:15-12:45pm

Cryptographic Hash Functions

- A **cryptograph hash function** is a function
$$\mathsf{Hash} : \begin{pmatrix}\text{arbitray length}\\ \text{bit strings}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{bit strings of a}\\ \text{specified length}\end{pmatrix}.$$
- They are crucial for modern encrypted communications.
- Required properties:
  - $\mathsf{Hash}()$ is easy to compute.
  - Given a specified output γ, it's hard to find an input β satisfying
$$\mathsf{Hash}(\beta) = \gamma.$$
  - It is hard to find distinct inputs $\beta_1 \neq \beta_2$ satisfying
$$\mathsf{Hash}(\beta_1) = \mathsf{Hash}(\beta_2).$$

Turning an Expander Graph into a Hash Function

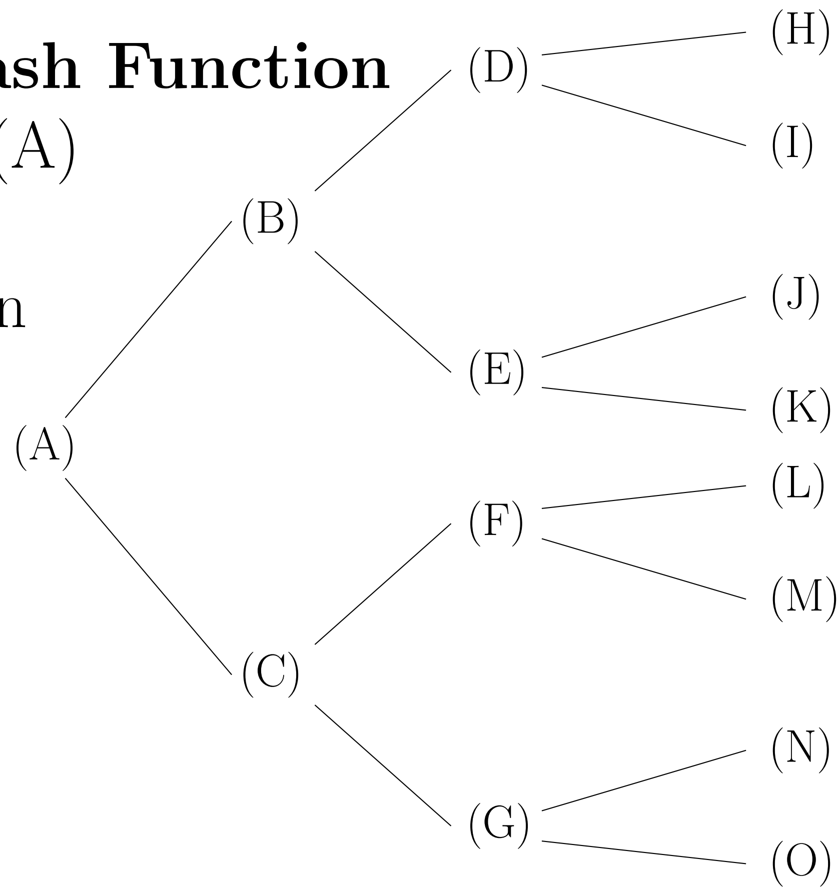
- Charles, Goren, and Lauter [*J. Cryptology* **22** (2009)] explained how to use expander graphs to construct hash functions, assuming that it is hard to find paths between specified initial and final vertices.

Graph to Hash Function

Rule: Start at (A)

1 = Up

0 = Down



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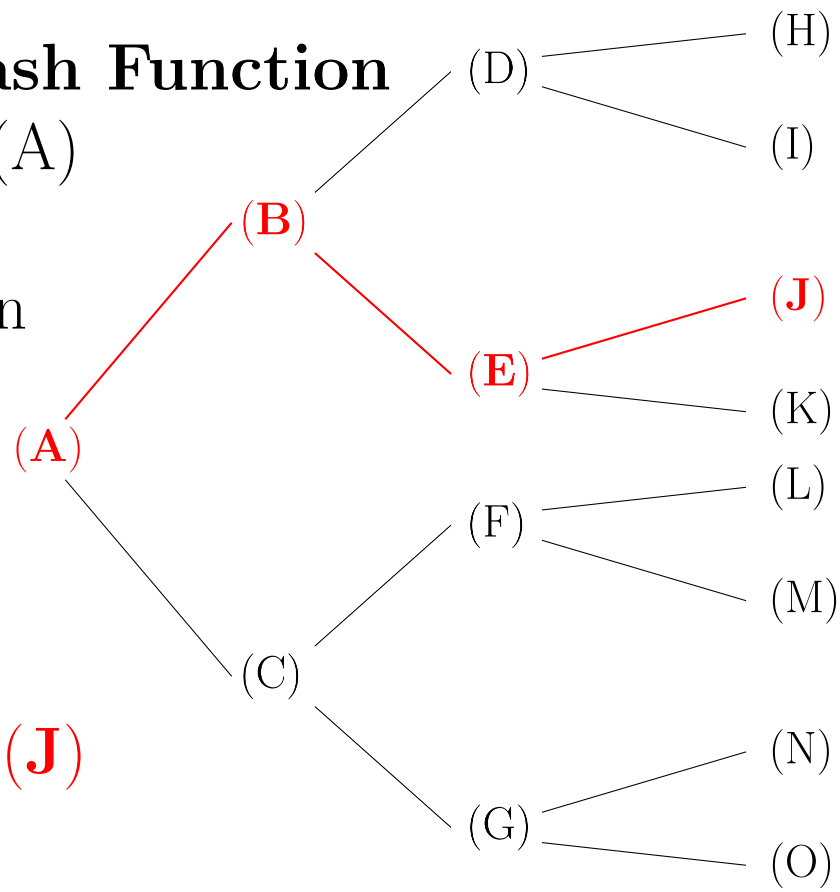

$\mathbf{101} \xrightarrow{\text{Hash}} \mathbf{(J)}$

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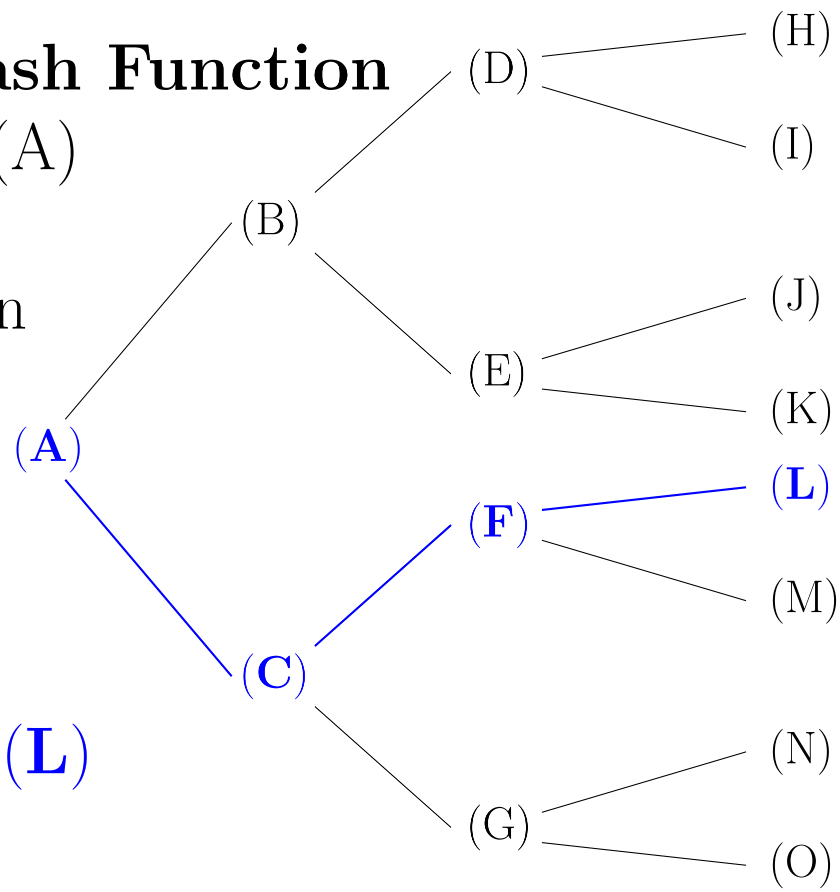
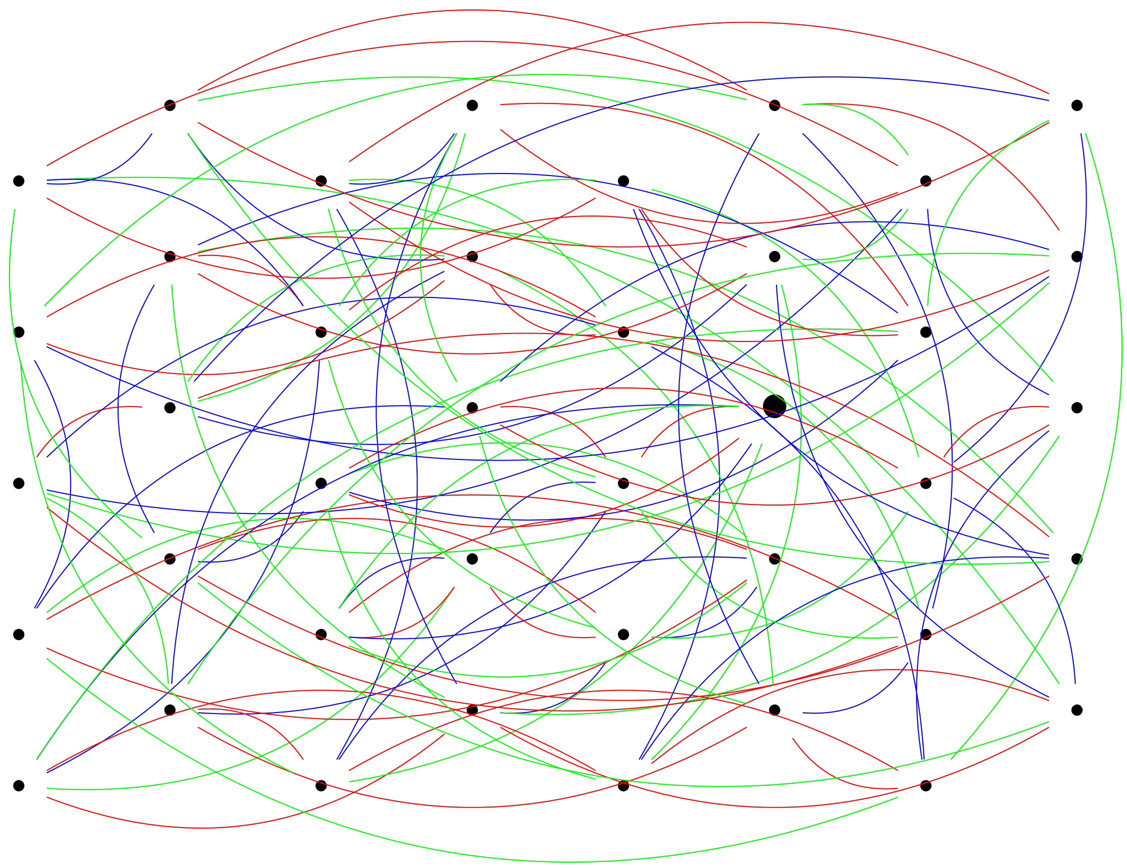


$\mathbf{011} \xrightarrow{\text{Hash}} (\mathbf{L})$

Turning an Expander Graph into a Hash Function

In practice one uses a large finite graph with a marked initial point.



The Markoff Equation

The **Markoff equation** is

$$\mathcal{M} : x^2 + y^2 + z^2 = 3xyz.$$

The equation is quadratic in each variable, so if we're given any solution (x_0, y_0, z_0), we can create a new solution by fixing two of the coordinates and switching the third coordinate to the other root of the quadratic equation.

This gives three non-commuting involutions

$$\sigma : \mathcal{M} \longrightarrow \mathcal{M},$$

and composing them with a coordinate permutation gives three *non-commuting rotations* given by the easily computed formulas

$$\begin{aligned} \rho_1(x, y, z) &= (x, z, 3xz - y), \\ \rho_2(x, y, z) &= (3xy - z, y, x), \\ \rho_3(x, y, z) &= (y, 3yz - x, z). \end{aligned}$$

The Markoff Graph

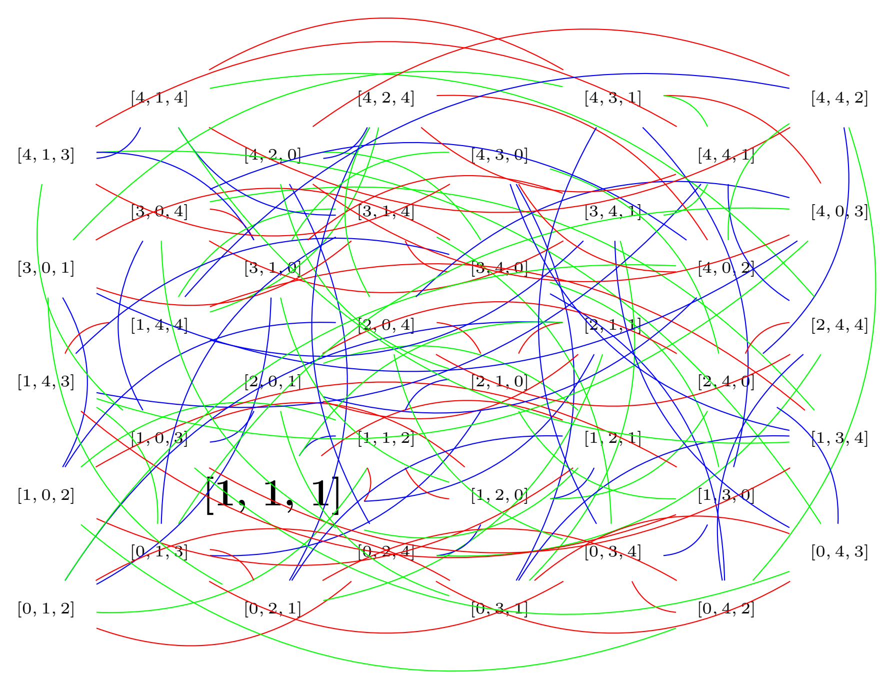
We use the set of non-zero points

$$\mathcal{M}(\mathbb{F}_p) = \left\{ \begin{matrix} \text{solutions to } x^2 + y^2 + z^2 = 3xyz \\ \text{with } x, y, z \text{ in the finite field } \mathbb{F}_p \end{matrix} \right\}$$

to create a graph with

$$\text{Vertices} = \mathcal{M}(\mathbb{F}_p), \quad \text{Initial Point} = [1,1,1],$$
$$\text{Edges} = \Big\{ \big[P, \rho_i(P)\big] : i = 1,2,3 \Big\}.$$

$\mathcal{M}(\mathbb{F}_5)$



Properties of the Markoff Graph

- $\mathcal{M}(\mathbb{F}_p)$ has roughly p^2 vertices. [Elementary]
- $\mathcal{M}(\mathbb{F}_p)$ is a connected graph for all sufficiently large p. [Bourgain–Gamburd–Sarnak, W. Chen]
- $\mathcal{M}(\mathbb{F}_p)$ is a family of expander graphs [Conjecture]

- Fuchs, Lauter, Litman, and Tran (2022) suggested that the Markoff graphs "may be good candidates" for the CGL hash function construction.
- In the remainder of this talk, I will sketch a heuristic path-finding algorithm for $\mathcal{M}(\mathbb{F}_p)$ that is subexponential time on a classical computer and polynomial time on a quantum computer.
- More precisely, to connect points in $\mathcal{M}(\mathbb{F}_p)$, it suffices to factor $p-1$ and to solve three discrete logarithm in $\mathbb{F}_p^*$.

Proof Sketch (as time permits)

We exploit ideas used by Bourgain–Gamburd–Sarnak. They note that for fixed x_0,

$$\rho_1(x_0, y, z) = \left[x_0, \begin{pmatrix} 3x_0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \right].$$

Thus ρ_1 acts on the $x = x_0$ fiber via the matrix

$$L_{x_0} := \begin{pmatrix} 3x_0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_p).$$

$$L_{x_0} \text{ has order } p-1 \quad \Longrightarrow \quad \begin{pmatrix} \rho_1 \text{ acts transitively} \\ \text{on the } x = x_0 \text{ fiber} \end{pmatrix}.$$

If this occurs, we say that x_0 is **maximally hyperbolic**. And similarly for ρ_2 and ρ_3.

For randomly chosen points in $\mathcal{M}(\mathbb{F}_p)$, we have

$$\mathrm{Prob}\begin{pmatrix} P \in \mathcal{M}(\mathbb{F}_p) \text{ is } x(P)\text{-} \\ \text{maximally hyperbolic} \end{pmatrix} \approx \frac{\phi(p-1)}{2(p-1)} \geq \frac{1}{4\log\log p}.$$

Finding a path from $P \in \mathcal{M}(\mathbb{F}_p)$ to $Q \in \mathcal{M}(\mathbb{F}_p)$

(1) Randomly apply ρ_1 and ρ_3 to P until reaching a point P' that is y-maximally hyperbolic.

(2) Randomly apply ρ_1^{-1} and ρ_2^{-1} to Q until reaching a point Q' that is z-maximally hyperbolic.

(3) Let $F(X,Y,Z) = X^2 + Y^2 + Z^2 - 3XYZ$. Randomly select maximally hyperbolic $x_0 \in \mathbb{F}_p$ until the pair of quadratic equations

$$F\bigl(x_0, y(P'), Z\bigr) = F\bigl(x_0, Y, z(Q')\bigr) = 0$$

has a solution $(y_0, z_0) \in \mathbb{F}_q^2$. Set

$$P'' \leftarrow \bigl(x_0, y(P'), z_0\bigr) \quad \text{and} \quad Q'' \leftarrow \bigl(x_0, y_0, z(Q')\bigr).$$

- P'' and Q'' are on the maximally hyperbolic x_0-fiber.
- P' and P'' are on the maximally hyperbolic $y(P')$-fiber.
- Q' and Q'' are on the maximally hyperbolic $z(Q')$-fiber.

Finding a path from $P \in \mathcal{M}(\mathbb{F}_p)$ to $Q \in \mathcal{M}(\mathbb{F}_p)$

(4) Solve three DLPs in $\mathbb{F}_p^*$ to find k, m, n satisfying

$$P'' = \rho_2^k(P'), \quad Q' = \rho_3^m(Q''), \quad Q'' = \rho_1^n(P'').$$

These are DLPs because maximal hyperbolicity means that the associated matrices diagonalize over $\mathbb{F}_p$, so we end up needing to solve equations of the form

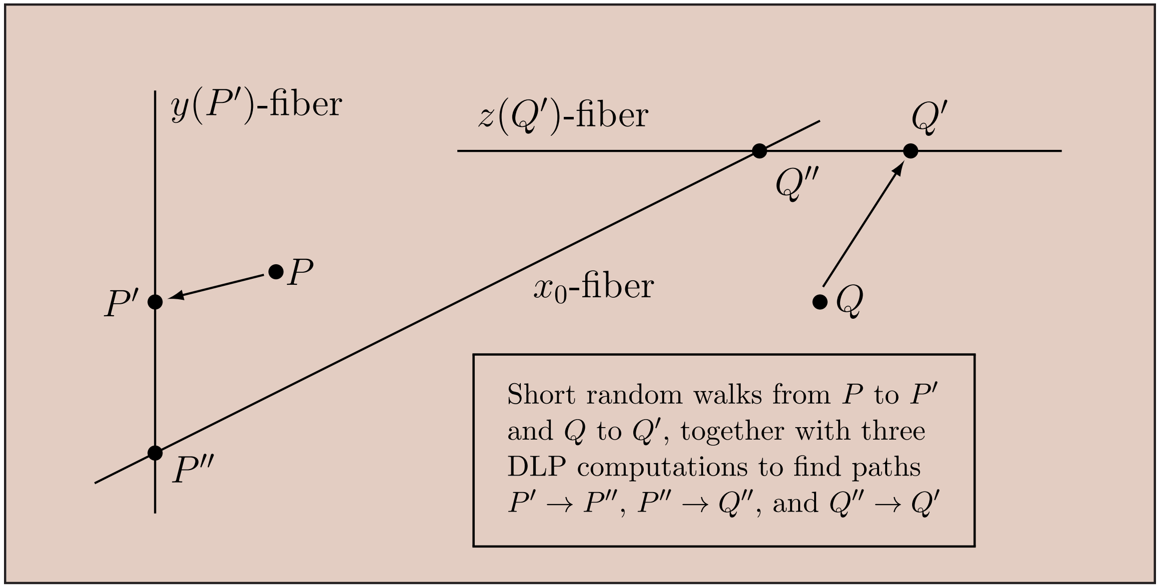
$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}^n \begin{pmatrix} \alpha \\ \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \beta \\ \beta^{-1} \end{pmatrix} \quad \text{for known } \lambda, \alpha, \beta.$$

Finding a path from $P \in \mathcal{M}(\mathbb{F}_p)$ to $Q \in \mathcal{M}(\mathbb{F}_p)$

(5) This gives the path

$$P \xrightarrow{\langle\rho_1,\rho_3\rangle} P' \xrightarrow{\rho_2^k} P'' \xrightarrow{\rho_1^n} Q'' \xrightarrow{\rho_3^m} Q' \xrightarrow{\langle\rho_1,\rho_2\rangle} Q.$$

Illustrating the Markoff Path-Finding Algorithm



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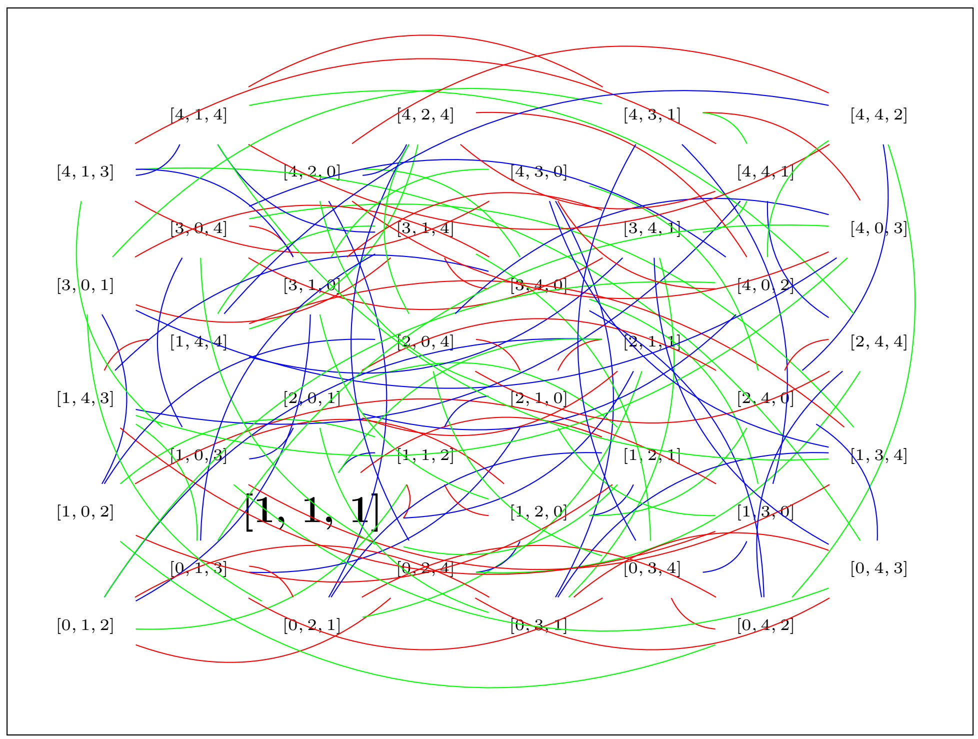

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