Computation of Fourier coefficients of automorphic forms of type G_2

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Classical modular forms

Suppose

1. $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ is a finite index subgroup,
2. $\ell \geq 0$ is an integer,
3. $\mathfrak{h} = \{z \in \mathbf{C} : Im(z) > 0\}$ is the complex upper half-plane.

Recall that $\mathrm{SL}_2(\mathbf{R})$ acts on $\mathfrak{h}$ by fractional linear transformations:

$$\gamma z = \frac{az+b}{cz+d} \text{ if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Modular form

A modular form of weight ℓ and level Γ is a function $f : \mathfrak{h} \to \mathbf{C}$ that satisfies

1. $f(\gamma z) = (cz+d)^{\ell} f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$
2. f is holomorphic
3. f has "moderate growth"

Fourier expansion

- Suppose (for simplicity) that $\Gamma \supseteq \{\left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right) : n \in \mathbf{Z}\}$.
- Then $f(z+1) = f(z)$,
- and f holomorphic, so
$$f(z) = \sum_{n \in \mathbf{Z}} a_f(n) e^{2\pi i n z} \text{ with } a_f(n) \in \mathbf{C}.$$

Moderate growth implies: $a_f(n) = 0$ if $n < 0$.

Fourier expansion

If f is a modular form,
$$f(z) = \sum_{n \geq 0} a_f(n) e^{2\pi i n z}.$$
The $a_f(n)$ are called the **Fourier coefficients** of f.

Computation of modular forms?

The modular forms $M_\ell(\Gamma)$ of weight ℓ and level Γ form a finite-dimensional vector space.

A question

Can one compute exactly (on a computer) many Fourier coefficients of a basis of $M_\ell(\Gamma)$?

- The answer requires both a theorem, and an implementation.
- The "theorem" in this case might be modular symbols, a trace formula, theta functions, something with multiplying modular forms.

Automorphic forms

Suppose G is a (non-compact) Lie group, and $\Gamma \subseteq G$ is an infinite discrete subgroup. Let $K \subseteq G$ be a maximal compact subgroup, and (V, ρ) a finite-dimensional representation of K.

A rough definition

A (cuspidal) automorphic form on G of weight ρ and level Γ is a smooth (bounded) function $\varphi : G \to V$ satisfying

1. $\varphi(gk) = \rho(k)^{-1}\varphi(g)$ for all $k \in K$;
2. $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \Gamma$;
3. φ satisfies some differential equation $D\varphi \equiv 0$ that comes from group theory.

Automorphic forms specialize to modular forms, by taking $G = \mathrm{SL}_2(\mathbf{R})$.

Motivating question

Can they be computed?

The motivating question for this work is if, in certain cases, the modular forms can be computed (in some sense to be made precise) on a computer.

1. To answer this question requires a way of representing modular forms as, for example, sequences of numbers. This is a theorem, like the existence of some kind of Fourier expansion
2. A theorem that allows one to compute these sequences of numbers in finite time
3. An implementation for Step 2.

In this talk, I focus on the case of $G = G_2$, the exceptional group.

Modular forms on G_2

- G_2: a **noncompact** simple Lie group of dimension 14
- $K = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mu_2$ is a maximal compact subgroup of G_2
- $\mathbf{V}_\ell := Sym^{2\ell}(\mathbf{C}^2) \boxtimes \mathbf{1}$, $\ell \geq 1$ integer, a representation of K.

Definition (Gross-Wallach, Gan-Gross-Savin)

Suppose $\Gamma \subseteq G_2$ a congruence subgroup. A cuspidal modular form on G_2 of weight ℓ and level Γ is a smooth, bounded function $\varphi : \Gamma \backslash G_2 \to \mathbf{V}_\ell$ satisfying

1. $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $k \in K$ and
2. $D_\ell \varphi \equiv 0$ for a certain special linear differential operator D_ℓ.

We denote the cuspidal forms of weight ℓ and level Γ by $S_\ell(\Gamma)$. They form a finite-dimensional complex vector space.

Modular forms on G_2 have a Fourier expansion

Theorem 1 (P. 2020)

Fix $\ell \geq 1$. For every real binary cubic form f, there is an explicit function $W_f : G_2 \rightarrow \mathbf{V}_\ell$. If φ a cuspidal modular form of weight ℓ and level Γ (sufficiently large), then

$$\varphi(g) \text{``}=\text{''} \sum_{\mathrm{Disc}(f)>0, f \text{ integral}} a_\varphi(f) W_f(g)$$

for some $a_\varphi(f) \in \mathbf{C}$.

Remark

The existence of the Fourier coefficients (without the explicit functions W_f) was given by Gan-Gross-Savin, crucially using a result of Wallach.

Fourier coefficients on G_2 are computable

Remark

The Fourier coefficients $a_\varphi(f)$ are defined in a very transcendental way. There is no *a priori* reason that they might be exactly computable, in even a single nonzero example.

Nevertheless, in a previous work:

Theorem 2 (P. 2023)

Suppose $\ell \geq 6$ is even. The Fourier coefficients of elements of elements of $S_\ell(\Gamma)$ can be computed in terms of vectors in finite-dimensional representations of the exceptional group F_4. There is a basis of $S_\ell(\Gamma)$ for which every Fourier coefficient of an element of this basis lies in $\mathbf{Q}^{cyc}$, the cyclotomic extension of $\mathbf{Q}$.

The content of the present work is to implement the formulas of this theorem and to get consequences of these calculations.

Theta functions work

1. **No modular symbols**: So far as has been proved, there is no real analogue of modular symbols to the case of G_2
2. **Not the trace formula**: The trace formula, even if you could make it explicit, would give you eigenvalues of Hecke operators. This is great information, but it doesn't tell you the Fourier coefficients.
3. **No multiplication**: You cannot multiply modular forms on G_2.
4. **Theta functions**: It turns out, there is an analogue of harmonic theta functions on G_2. Theorem 2 on the previous page is the combination of two separate theorems:
    1. (P.) The existence of an analogue to G_2 of harmonic theta functions: cuspidal modular forms, whose Fourier coefficients can be written down concretely.
    2. (P.) A theorem that says that every cuspidal QMF (of even weight at least 6) is one of these harmonic theta functions.

Theta functions, quickly

The data

1. J, the $27 = 1 + 26$-dimensional real-representation of F_4.
2. $\Lambda \subseteq J$ a lattice.
3. Special points $I, E \in \Lambda$ (Elkies-Gross)
4. Certain allowable elements $X \wedge Y \in \wedge^2 J \otimes \mathbf{C}$.
5. An integer $m \geq 1$.

The output

1. Cusp forms $\Theta_{I,m}(X;Y)(g)$, $\Theta_{E,m}(X,Y)(g)$ on G_2 of weight $4+m$
2. To each $T \in \Lambda$, there are two associated monic binary cubic forms $f_{T,I}(u,v) = u^3 + b_I(T)u^2v + c_I(T)uv^2 + d(T)v^3$ and similarly $f_{T,E}(u,v)$.
3. A polynomial $Q_I(T;X,Y)$ (cubic in T, linear in X and Y).
4. The $f(u,v)$ Fourier coefficient of $\Theta_{I,m}$ is a sum of $Q_I(T;X,Y)^m$ for all T with $f_{T,I} = f$.

Dimension formula

A result of Rahul Dalal

In case $\Gamma = G_2(\mathbf{Z})$, Rahul Dalal has computed $\dim_\ell S_\ell(G_2(\mathbf{Z}))$ explicitly, using the Arthur-Selberg trace formula.

1. The smallest nonzero example, $F_6(g)$, occurs in weight $\ell = 6$. The space $S_6(G_2(\mathbf{Z}))$ is one-dimensional.
2. The spaces $S_9(G_2(\mathbf{Z}))$ and $S_{11}(G_2(\mathbf{Z}))$ are also one-dimensional. Let $F_9(g)$, $F_{11}(g)$ denote basis elements.
3. It follows from Dalal's dimension formula that $\dim S_\ell(G_2(\mathbf{Z}))$ grows like ℓ^5.

An example

In the table, the ordered 3-tuple is the (b, c, d) of the monic binary cubic $f(u, v) = u^3 + bu^2v + cu^2v + dv^3$, and $a(f)$ is the associated Fourier coefficient of $F_6(g)$.

$f(u, v) = u^3 + bu^2v + cu^2v + dv^3$	$a(f)$
$(0, -3, -1)$	48600
$(0, -3, 0)$	1620
$(0, -2, -1)$	15
$(0, -2, 0)$	1680
$(0, -1, 0)$	-7
$(1, -3, -3)$	-10080
$(1, -3, -2)$	25575
$(1, -3, -1)$	28800
$(1, -3, 0)$	-1485
$(1, -2, -2)$	-30
$(1, -2, -1)$	12600
$(1, -2, 0)$	-63

A conjecture of Gross

1. The Arthur Multiplicity Conjecture predicts that certain special cusp forms on G_2 of weight ℓ "come from" classical cusp forms of weight 2ℓ.
2. Associated to an $\mathrm{SL}_2(\mathbf{Z})$-orbit of an integral binary cubic form f is a cubic ring R_f. We write $a_\varphi(R) := a_\varphi(f)$ if $R = R_f$. (The Fourier coefficients of $\varphi \in S_\ell(G_2(\mathbf{Z}))$ are constant on $\mathrm{SL}_2(\mathbf{Z})$-orbits of integral binary cubics.)

A conjecture of Gross

Suppose $\alpha \in S_{2\ell}(\mathrm{SL}_2(\mathbf{Z}))$ is an eigenform, and $F_\alpha \in S_\ell(G_2(\mathbf{Z}))$ is its lift, as predicted by Arthur. Suppose E is a totally real cubic etale algebra, R is the maximal order in E, and let V_E be the two-dimensional motive satisfying $\zeta_E(s) = \zeta_{\mathbf{Q}}(s)L(V_E, s)$. The square Fourier coefficients $a_{F_\alpha}(R)^2$ should be related to the value of the L-function $L(\alpha \times V_E, s)$ at its center point.

Results on F_9 and F_{11}

The space of cusp form $S_{18}(\mathrm{SL}_2(\mathbf{Z}))$ and $S_{22}(\mathrm{SL}_2(\mathbf{Z}))$ are one-dimensional.

- If Arthur is correct, the cusp forms F_9, F_{11} on G_2 should be lifts of these SL_2 cusp forms.

Suppose F is a real quadratic field, and $E = \mathbf{Q} \times F$. Let $\alpha \in S_{2\ell}(\mathrm{SL}_2(\mathbf{Z}))$ be an eigenform, with ℓ odd. Then $L(\alpha \times V_E, s)$ vanishes at the center.

- If Gross' conjecture is correct, we must have $a_\varphi(\mathbf{Z} \times \mathcal{O}_F) = 0$ for $\varphi = F_9, F_{11}$.

Theorem 3

For all real quadratic F, one has $a_\varphi(\mathbf{Z} \times \mathcal{O}_F) = 0$ for $\varphi = F_9, F_{11}$.

Proof of Theorem 3

Proof.

1. By computer calculation, there are X, Y allowable so that $\Theta_{E,m}(X,Y)(g) \neq 0$, for $m = 5, 7$.
2. By one-dimensionality (Dalal), $\Theta_{E,5}(X,Y) = F_9$, $\Theta_{E,7}(X,Y) = F_{11}$.
3. Let $R = \mathbf{Z} \times \mathcal{O}_F$ and $f_R(u,v)$ an associated binary cubic.
4. Properties (Elkies-Gross) of the pointed lattice (Λ, E) imply that the sum giving the Fourier coefficient of $\Theta_{E,m}(X,Y)$ corresponding to $f_R(u,v)$ is empty.
5. The property in (4) is special to (Λ, E); it does not hold for (Λ, I).

□

Thank you

Thank you for your attention!