Solving Norm Equations in Global Function Fields

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Outline

Motivation

- Solving Diophantine equations is a difficult problem as well as a classical problem in Mathematics.
- Norm equations are a special type of Diophantine equations over global fields; number fields and global function fields.
- Solving norm equations over global function fields has not been studied as much as it over number fields has.
- To solve norm equations over global function fields using the existing algorithm by Gaál and Pohst, we need to enumerate a huge number of elements, and the elements may be of huge sizes.
- Using shorter representations helps to practically solve the norm equations.

Summary of Contributions

- Developed two new algorithms for solving norm equations using compact representation; one exhaustive search algorithm and one algorithm via index calculus (principal ideal tests).
- Thoroughly analyzed asymptotic complexity of the existing algorithms; Gaál-Pohst algorithm for solving norm equations and Eisentrager-Hallgren algorithm for computing compact representations.
- Performed complexity analysis of the new algorithms.
- Implemented and tested all three algorithms for solving norm equations (Gaál-Pohst and two new algorithms), and Eisentrager-Hallgren algorithm for computing compact representations. The algorithms were tested in different parameters and compared the test results to their complexity.
- The new algorithms were exponentially faster than the Gaál-Pohst algorithm.

Norm equation over a global function field F

Let $F/\mathbb{F}_q(x)$ be a finite extension of degree n by an irreducible defining polynomial $f(t) \in \mathbb{F}_q[x][t]$. The finite maximal order of F is denoted by O_F.

Norm of an element $\alpha \in F$

The norm of an element $\alpha \in F$ is defined as

$$\mathrm{Norm}_{F/\mathbb{F}_q(x)}(\alpha) = \prod_{j=1}^{n} \sigma_j(\alpha) \quad \in \mathbb{F}_q(x)$$

where σ_j are embeddings of F into the algebraic closure $\overline{\mathbb{F}_q(x)}$ of $\mathbb{F}_q(x)$.

Norm equation over a global function field F

Norm equation over F

A norm equation over F is defined as

$$\mathrm{Norm}_{F/\mathbb{F}_q(x)}(\alpha) = c,$$

where $c \in \mathbb{F}_q[x]$, and $\alpha \in O_F$.

- Note that if $\mathrm{Norm}_{F/\mathbb{F}_q(x)}(\alpha) = c$ for some $\alpha \in O_F$, then for any unit ε of O_F, $\mathrm{Norm}_{F/\mathbb{F}_q(x)}(\alpha\varepsilon) = c\zeta$ for some $\zeta \in \mathbb{F}_q^*$. We call $\alpha\varepsilon$ and α are associate.
- Solving a norm equation means, for a given $c \in \mathbb{F}_q[x]$, finding all non-associate α in the finite maximal order O_F of F that satisfy $\mathrm{Norm}_{F/\mathbb{F}_q(x)}(\alpha) = \zeta c$ for some $\zeta \in \mathbb{F}_q^*$.
- There was one method available to solve norm equations over F by Gaál and Pohst.

Standard representation of $\alpha \in F$

Let $F/\mathbb{F}_q(x)$ be a finite extension of fields of degree n, O_F be the finite maximal order of F, and $\mathcal{B}$ be an integral basis of F.

Standard representation of $\alpha \in F$

Let $\alpha \in F$. The standard representation of α with respect to an integral basis $\mathcal{B} = \{b_i \mid 1 \leq i \leq n\}$ is

$$\alpha = \sum_{i=1}^{n} a_i b_i,$$

where the coefficients $a_i \in \mathbb{F}_q(x)$.

- When α is in O_F, a_i are in $\mathbb{F}_q[x]$.
- $\deg a_i = O(n2^{n^2/4}q^g + \frac{\deg c}{n})$ when $\mathrm{Norm}_{F/\mathbb{F}_q(x)}(\alpha) = c$.

Compact representation of $\alpha \in F$

Compact representation is an alternative way to represent elements of F.

Compact representation

Let $\mathbb{P}_\infty = \{P_1, \ldots, P_{|\mathbb{P}_\infty|}\}$ be the set of the infinite places of F. For $\alpha \in F$, let $val_\infty(\alpha) = [v_{P_1}(\alpha) \ \ldots \ v_{P_{|\mathbb{P}_\infty|-1}}(\alpha)]$. An element $\alpha \in F$ can be written as a power product,

$$\alpha = \mu \prod_{i=1}^{l} \left(\frac{1}{\beta_i}\right)^{2^{l-i}}$$

where $l = \lfloor \log \| val_\infty(\alpha) \|_\infty \rceil + 1$, μ, $\beta_i \in F$. Thus, a vector

$$\mathbf{t}_\alpha = (\mu, \beta_1, \beta_2, \cdots, \beta_l) \in F^{l+1}$$

can represent α, and $\mathbf{t}_\alpha$ is called a compact representation of α.

Size of the compact representation of a global function field

The size of a representation is defined by the number of bits needed to store the representation.

Let $F = \mathbb{F}_q(x)(y)$ be a global function field of degree n and genus g.

- For a solution of the norm equation $\mathrm{Norm}_{F/\mathbb{F}_q(x)}(\alpha) = c$, the size of μ is $O(n \deg c + g)$, and the size of each β_i is $O(n^2 + ng)$.

Solving norm equation

The Gaál-Pohst algorithm finds solutions $\alpha = \sum_{i=1}^{n} a_i b_i$ of a norm equation in standard representation by the exhaustive search method that is

- using the search space defined by bounds on $\deg a_i$,
- checking the norm of each element in the search space,

and collecting α satisfies $\mathrm{Norm}_{F/\mathbb{F}_q(x)}(\alpha) = \zeta c$, $\zeta \in \mathbb{F}_q^*$.

That means,

- the search space contains a huge number of elements, which is doubly exponential in n and g, and
- we need to compute the norm of each element in the search space.
- Since we have $\deg a_i = O(n2^{n^2/4}q^g + \frac{\deg c}{n})$, the number of elements in the search space $O(q^{n^2 2^{n^2/4} q^g + \frac{\deg c}{n}})$.

Example

Let $F/\mathbb{F}_5(x)$ be a finite extension of degree $n = 3$ by

$$f(t) = t^3 + (4x^3 + 3x^2 + 1)t^2 + (3x^3 + 4x^2 + 4x + 2)t + 2x^3 + x.$$

Then the genus of F is 4, and the ideal class number of F is 2. There are two infinite places $\{P_{\infty,1}, P_{\infty,2}\}$ of F, with ramification indices $e_{P_{\infty,1}|p_\infty} = e_{P_{\infty,2}|p_\infty} = 1$. So F has unit rank 1.

Let $c = x + 4$. Consider a norm equation

$$\mathrm{Norm}_{F/\mathbb{F}_5(x)}(\alpha) = c(= x + 4).$$

To solve the norm equation, Gaál-Pohst algorithm enumerates 5^{1038} elements and checks their norms. The server (Intel Xeon CPU E7-8891 v4 with 80 64-bit cores at 2.80GHz) could not solve this for over 4 days.

Two new algorithms for solving norm equations

The first new algorithm is an exhaustive search algorithm.

- Its search space is defined by the values at finite and infinite places.
- The algorithm enumerates elements in compact representation, and computes the norms using compact representations.

The second new algorithm is via principal ideal tests.

- Finding $\alpha \in O_F$ such that $\mathrm{Norm}_{F/k(x)}(\alpha) = c$ is equivalent to finding principal ideals of the norm c.
- It is sufficient to search ideals that divide the principal ideal cO_F generated by c to find all principal ideals of norm c.
- For each ideal, we compute the norm of ideals. For each ideal of norm c up to a constant, we perform a principal ideal test and compute a generator in compact representation.

Benefits of using compact representations for solving norm equations

Using compact representations helps reduce

- the storage space to represent solutions,
- the number of elements in the search space to $O(2^{n^3/4+n(\deg c+gq^{\varepsilon})})$ instead of $O(q^{n^2 2^{n^2/4} q^g + \frac{\deg c}{n}})$, and
- the cost of computing the norm of each element in the search space to $O(n^{5+\varepsilon} 2^g (\deg c)^{1+\varepsilon} q^{\varepsilon})$ instead of $O((n^{4+\epsilon} 2^{n^2/4} q^g + \frac{\deg c}{n}) q^{\varepsilon})$.

Example I

Consider the same F and c, let $F/\mathbb{F}_5(x)$ be a finite extension of degree $n = 3$ by

$$f(t) = t^3 + (4x^3 + 3x^2 + 1)t^2 + (3x^3 + 4x^2 + 4x + 2)t + 2x^3 + x,$$

and $c = x + 4$. Consider the same norm equation

$$\mathrm{Norm}_{F/\mathbb{F}_5(x)}(\alpha) = c(= x + 4).$$

Using compact representations, our first new algorithm only needed to search $2 \cdot 2 \cdot (347 \cdot 2 + 1) = 2980$ (approximately $5^{4.97}$) elements which is significantly less than 5^{1038}. It was solved in 114.830 CPU seconds, and there was one solution, $\mathbf{t} = (\mu, \beta_1, \beta_2, \ldots, \beta_9)$ where

Example II

$$
\begin{aligned}
\mu =& y + 4,\\
\beta_1 =& \beta_2 = \beta_9 = 1, \;\; \beta_3 = 4,\\
\beta_4 =& (3x^2 + 3x)y^2 + (2x^5 + x^4 + 4x^3 + 4x^2 + 3x + 1)y + 3x^5 + 4x^4 + 3x^3 + x^2 + 3x + 2,\\
\beta_5 =& \frac{3x^5 + 4x^4 + 3x^3 + 2x^2 + 3x + 1}{x^8 + 2x^7 + 2x^6 + 4x^5 + x^4 + 3x^3 + 3x^2 + x + 4} y^2\\
&+ \frac{2x^8 + 4x^6 + 3x^5 + x^4 + 2x^3 + 3x + 1}{x^8 + 2x^7 + 2x^6 + 4x^5 + x^4 + 3x^3 + 3x^2 + x + 4} y\\
&+ \frac{x^8 + 4x^7 + 3x^6 + x^5 + 2x^4 + x^3 + 4x^2 + 4x + 3}{x^8 + 2x^7 + 2x^6 + 4x^5 + x^4 + 3x^3 + 3x^2 + x + 4},\\
\beta_6 =& \frac{x^4 + 2x^3 + 2x^2 + 2}{x^7 + 2x^5 + 3x^4 + 2x^2 + x + 1} y^2 + \frac{4x^7 + x^6 + 4x^5 + 2x^4 + 3x^3 + x^2 + 2x + 1}{x^7 + 2x^5 + 3x^4 + 2x^2 + x + 1} y\\
&+ \frac{3x^7 + 2x^6 + 4x^5 + 2x^4 + 2x^2 + 2x + 4}{x^7 + 2x^5 + 3x^4 + 2x^2 + x + 1},
\end{aligned}
$$

Example III

$$\beta_7 = \frac{2x^4 + x^3 + x^2 + x + 4}{x^8 + 2x^7 + x^6 + x^4 + x^3 + 4} y^2 + \frac{3x^7 + 3x^5 + 2x^3 + 4x + 4}{x^8 + 2x^7 + x^6 + x^4 + x^3 + 4} y + \frac{4x^8 + 3x^7 + 2x^6 + 3x^5 + x^4 + x^3 + 2x^2 + 3x + 4}{x^8 + 2x^7 + x^6 + x^4 + x^3 + 4},$$

$$\beta_8 = \frac{4x^4 + 3x^3 + 2x^2 + 3x + 2}{x^8 + 4x^7 + 3x^6 + 2x^4 + 2x^3 + 4x^2 + 4x + 4} y^2 + \frac{x^5 + 4x^4 + 4x^3 + x^2 + 3}{x^6 + 4x^5 + 3x^3 + 2x^2 + 3x + 3} y + \frac{x^5 + x^4 + 4x^3 + 3x^2 + 3x + 3}{x^6 + 4x^5 + 3x^3 + 2x^2 + 3x + 3}.$$

Benefits of performing principal ideal tests to solve norm equations

- Searching ideals instead of elements and using compact representations to represent solutions decrease the asymptotic complexity significantly, because the number of ideals is $O(2^{n \deg c})$ which is much less than the number of elements, $O(q^{n^2 2^{n^2/4} q^g + \frac{\deg c}{n}})$ and $O(2^{n^3/4 + n(\deg c + g q^{\varepsilon})})$.
- The cost of computing the norm of an ideal is $O(n^{2+\varepsilon} \deg(c)^{1+\varepsilon} q^{\varepsilon})$ which is significantly smaller than the costs of computing the norm of an element in standard representation $O((n^{4+\epsilon} 2^{n^2/4} q^g + \frac{\deg c}{n}) q^{\varepsilon})$ and in compact representation $O(n^{5+\varepsilon} 2^g (\deg c)^{1+\varepsilon} q^{\varepsilon})$.

Example

Consider the same F and c, let $F/\mathbb{F}_5(x)$ be a finite extension of degree $n = 3$ by

$$f(t) = t^3 + (4x^3 + 3x^2 + 1)t^2 + (3x^3 + 4x^2 + 4x + 2)t + 2x^3 + x,$$

and $c = x + 4$. Consider the same norm equation

$$\mathrm{Norm}_{F/\mathbb{F}_5(x)}(\alpha) = c(= x + 4).$$

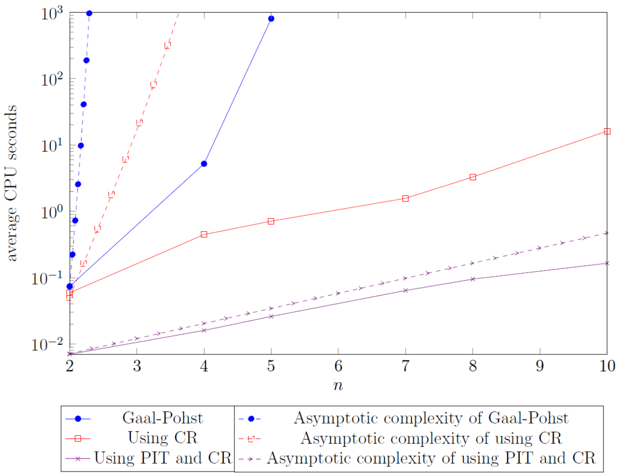
Using principal ideal tests and compact representations, our second new algorithm only needed to search 4 ideals which is significantly less than $5^{4.97}$ and 5^{1038}. It was solved in 0.180 CPU seconds.

Asymptotic complexity

Gaál-Pohst	Using CR	Using PIT and CR
$n \to \infty$		
$2^{O(n^2 2^{n^2/4})}$	$2^{n^3(1+o(1))}$	$2^{n(\deg c + o(1))}$
$n \to \infty$, c, cO_F irreducible		
$2^{O(n^2 2^{n^2/4})}$	$2^{n^3(1+o(1))}$	$O(2^{0.3774n})$
$g \to \infty$		
$2^{O(q^g)}$	$2^{O(g)}$	$O(g^{4+\varepsilon})$
$\deg c \to \infty$		
$O\left(2^{q^\varepsilon \deg c} (\deg c)^\omega\right)$	$O(2^{n \deg c} (\deg c)^\omega)$	$O(2^{n \deg c} (\deg c)^{\omega+2+\varepsilon})$
$\deg c \to \infty$, c irreducible		
$O\left(2^{q^\varepsilon \deg c} (\deg c)^\omega\right)$	$O((\deg c)^\omega)$	$O((\deg c)^2)$
$q \to \infty$		
$q^{O(q^g)}$	$O(q^{gn+\varepsilon})$	$O(q^\varepsilon)$

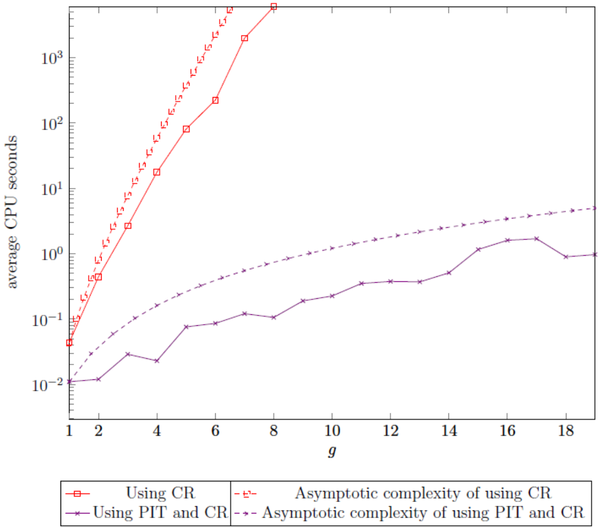
- We denoted an arbitrary power of $\log q$ as q^ε for brevity.
- ω is the matrix multiplication exponent. The best known bound on ω to date is $\omega < 2.37286$.

Testing results - varying n



Empirical timing results and asymptotic complexities for varying n
($g = 1$, $q = 3$, $h_{O_F} = 1$, $\deg c = 1$, and irreducible c)

Testing results - varying g



Empirical timing results and asymptotic complexities for varying g

Conclusion and open problems

- The complexity analysis and testing results provided evidence of the new algorithms' efficiency.
- Both new algorithms were exponentially faster in n and g compared to the existing algorithm of Gaál and Pohst.
- The second new algorithm was the fastest among three algorithms in all testing examples.

- Open problems:
  - Consider compact representations using cube-and-multiply instead of square-and-multiply, and compare the time and space efficiency with the existing ones.
  - Solving norm equations in submodules of O_F, which can be reduced to solving S-unit equations and Thue equations.

Thank you!

Example of Standard representation I

Let $F = \mathbb{F}_5(x)(y)$ where y is a root of the polynomial $f(t) = t^3 + (x^3+1)t^2 + (x^5+x^4+1)t + 2 \in \mathbb{F}_5[x][t]$ with
$O_F = \mathbb{F}_5[x] + y\mathbb{F}_5[x] + y^2\mathbb{F}_5[x]$,
$\mathcal{B} = \{1, y, y^2\}$.
A fundamental unit ϵ of F in the standard representation is:

$a_0 + a_1 y + a_2 y^2$ with

$a_0 = 2x^{312}+x^{311}+2x^{310}+x^{308}+4x^{306}+2x^{305}+x^{304}+3x^{303}+2x^{302}+3x^{301}+3x^{300}+$
$3x^{297}+4x^{295}+x^{294}+3x^{292}+x^{291}+3x^{290}+2x^{289}+2x^{288}+4x^{287}+x^{286}+3x^{285}+4x^{284}+$
$4x^{283}+3x^{282}+x^{281}+3x^{279}+2x^{278}+3x^{277}+2x^{276}+4x^{274}+x^{272}+4x^{271}+2x^{270}+2x^{269}+$
$2x^{267}+4x^{266}+4x^{265}+4x^{264}+x^{263}+2x^{262}+x^{260}+x^{257}+3x^{256}+3x^{255}+2x^{254}+4x^{253}+$
$2x^{252}+3x^{251}+3x^{250}+x^{248}+4x^{247}+2x^{246}+3x^{245}+2x^{243}+2x^{242}+4x^{241}+4x^{240}+4x^{237}+$
$x^{234}+3x^{233}+2x^{232}+x^{231}+3x^{228}+x^{227}+4x^{226}+3x^{224}+3x^{223}+2x^{222}+4x^{221}+x^{219}+$
$3x^{217}+3x^{216}+x^{215}+3x^{214}+3x^{210}+2x^{209}+x^{207}+4x^{205}+4x^{203}+3x^{202}+x^{201}+3x^{199}+$

Example of Standard representation II

$$x^{198}+2x^{197}+4x^{196}+x^{195}+4x^{194}+2x^{193}+3x^{192}+4x^{191}+4x^{190}+x^{189}+2x^{188}+2x^{186}+$$
$$4x^{185}+4x^{184}+2x^{181}+x^{180}+x^{179}+3x^{178}+2x^{177}+2x^{176}+4x^{173}+3x^{172}+x^{171}+4x^{170}+$$
$$x^{169}+3x^{168}+4x^{167}+3x^{166}+4x^{161}+2x^{159}+x^{158}+4x^{157}+x^{155}+3x^{154}+4x^{153}+2x^{152}+$$
$$x^{151}+x^{150}+2x^{149}+2x^{148}+2x^{147}+2x^{146}+4x^{145}+4x^{143}+2x^{141}+2x^{139}+x^{138}+4x^{137}+$$
$$4x^{135}+3x^{133}+3x^{132}+2x^{131}+4x^{129}+3x^{128}+4x^{127}+2x^{126}+4x^{125}+x^{122}+4x^{121}+3x^{120}+$$
$$3x^{119}+2x^{118}+3x^{115}+4x^{114}+x^{113}+x^{110}+x^{109}+4x^{108}+3x^{107}+2x^{106}+x^{105}+x^{104}+$$
$$2x^{103}+3x^{100}+4x^{99}+2x^{98}+3x^{97}+2x^{96}+4x^{95}+x^{94}+2x^{92}+2x^{91}+x^{90}+4x^{89}+3x^{86}+$$
$$3x^{85}+4x^{83}+2x^{82}+4x^{81}+2x^{80}+3x^{79}+2x^{77}+2x^{76}+x^{75}+3x^{74}+2x^{73}+x^{71}+2x^{70}+x^{69}+$$
$$4x^{68}+4x^{67}+x^{66}+x^{64}+x^{62}+3x^{61}+2x^{59}+3x^{58}+4x^{56}+x^{55}+x^{53}+2x^{52}+4x^{51}+x^{49}+$$
$$4x^{47}+x^{46}+3x^{45}+4x^{44}+x^{43}+4x^{42}+3x^{41}+x^{40}+4x^{39}+2x^{37}+x^{36}+x^{34}+3x^{33}+x^{31}+$$
$$x^{30}+4x^{29}+x^{27}+3x^{26}+x^{25}+3x^{24}+x^{23}+x^{22}+3x^{21}+4x^{19}+2x^{18}+4x^{17}+x^{16}+x^{15}+$$
$$4x^{14}+2x^{13}+4x^{12}+x^{11}+3x^{10}+2x^{9}+2x^{8}+2x^{7}+4x^{6}+2x^{5}+2x^{4}+4x^{3}+2x^{2}+2x+4,$$

Example of Standard representation III

$$a_1 = x^{317} + 4x^{316} + 4x^{315} + x^{314} + 3x^{313} + 4x^{312} + 4x^{310} + 2x^{309} + 2x^{308} + x^{307} + x^{306} + 3x^{305} + 2x^{304} + x^{302} + x^{301} + 3x^{300} + 2x^{299} + 4x^{298} + 2x^{297} + 2x^{295} + 2x^{294} + 3x^{293} + 4x^{292} + x^{291} + 3x^{290} + 3x^{289} + 4x^{288} + 3x^{287} + x^{286} + 4x^{283} + 3x^{282} + 4x^{280} + 2x^{276} + x^{275} + 3x^{274} + 3x^{273} + 2x^{272} + x^{271} + 4x^{270} + x^{269} + 4x^{268} + 2x^{267} + 3x^{266} + x^{264} + 3x^{263} + 4x^{262} + 4x^{261} + 2x^{260} + 3x^{259} + x^{258} + x^{257} + x^{256} + 4x^{255} + x^{254} + 3x^{253} + 2x^{250} + 4x^{249} + 4x^{248} + 4x^{247} + 3x^{245} + 4x^{244} + x^{243} + 2x^{242} + x^{241} + 4x^{240} + x^{238} + x^{237} + 3x^{236} + 3x^{235} + x^{233} + 4x^{232} + 3x^{231} + 3x^{228} + 3x^{227} + x^{226} + 4x^{225} + x^{224} + 4x^{223} + 4x^{222} + 3x^{221} + 2x^{220} + 2x^{219} + 4x^{218} + 2x^{217} + x^{216} + 2x^{214} + 4x^{213} + 4x^{212} + 2x^{211} + 3x^{210} + x^{209} + 4x^{208} + 3x^{205} + x^{202} + 2x^{201} + 4x^{198} + 2x^{197} + 4x^{196} + 4x^{195} + 2x^{194} + x^{193} + 4x^{192} + 3x^{190} + 3x^{189} + 2x^{188} + 2x^{187} + 2x^{186} + 4x^{184} + 4x^{183} + 3x^{182} + x^{181} + 4x^{180} + x^{179} + 4x^{178} + x^{177} + x^{176} + 3x^{174} + x^{173} + 3x^{171} + 4x^{170} + 2x^{169} + 2x^{166} + 4x^{165} + x^{163} + 4x^{162} + 4x^{161} + 2x^{160} + 4x^{159} + x^{158} + 3x^{157} + 4x^{156} + 2x^{155} + 2x^{154} + 3x^{153} + 2x^{152} + 3x^{150} + 4x^{148} + 3x^{147} + x^{146} + 3x^{145} + 3x^{144} + 4x^{143} + x^{142} + 4x^{141} + 3x^{140} + 2x^{139} + x^{138} + x^{136} + x^{135} + 2x^{134} + x^{133} + 4x^{132} + x^{131} + x^{130} + x^{127} + 3x^{126} + 3x^{124} + x^{122} + 4x^{120} + x^{119} + 2x^{118} + 4x^{117} + 2x^{114} + 4x^{113} + 2x^{111} + x^{110} + 3x^{109} +$$

Example of Standard representation IV

$$3x^{108}+3x^{105}+x^{103}+x^{100}+3x^{97}+3x^{96}+3x^{95}+4x^{94}+4x^{93}+4x^{92}+2x^{91}+3x^{90}+3x^{89}+x^{88}+2x^{87}+2x^{86}+2x^{85}+4x^{84}+x^{83}+4x^{81}+x^{80}+x^{79}+4x^{78}+2x^{77}+3x^{76}+x^{75}+3x^{74}+3x^{73}+2x^{72}+x^{69}+4x^{68}+4x^{67}+4x^{66}+x^{65}+3x^{64}+4x^{63}+2x^{62}+4x^{61}+2x^{60}+4x^{59}+3x^{58}+3x^{57}+4x^{56}+x^{55}+4x^{54}+x^{53}+3x^{51}+3x^{50}+4x^{48}+x^{46}+3x^{45}+4x^{44}+x^{42}+3x^{40}+4x^{39}+2x^{38}+x^{37}+3x^{36}+2x^{35}+4x^{33}+4x^{32}+3x^{30}+x^{29}+4x^{26}+3x^{25}+x^{22}+2x^{21}+3x^{19}+4x^{18}+2x^{17}+x^{16}+x^{15}+4x^{14}+x^{12}+3x^{11}+2x^{10}+4x^{9}+x^{8}+4x^{7}+2x^{5}+x^{3}+3x+4,$$

Example of Standard representation V

$$a_2 = x^{315} + 2x^{314} + x^{313} + x^{311} + 3x^{309} + 3x^{308} + 2x^{307} + 2x^{306} + 4x^{305} + 2x^{304} + 3x^{302} + 4x^{301} + 2x^{299} + 3x^{298} + 3x^{296} + 2x^{295} + 3x^{294} + 2x^{292} + 3x^{291} + x^{290} + 2x^{289} + 4x^{288} + 3x^{287} + 4x^{286} + 4x^{285} + 4x^{283} + x^{282} + 4x^{281} + 4x^{279} + 4x^{278} + 4x^{276} + 2x^{275} + 3x^{274} + x^{273} + 3x^{272} + 4x^{271} + 3x^{268} + 4x^{267} + 3x^{265} + x^{264} + 4x^{263} + x^{260} + 3x^{259} + 3x^{258} + 2x^{256} + 2x^{255} + x^{254} + x^{253} + 2x^{252} + 4x^{250} + 4x^{249} + x^{248} + 2x^{247} + 2x^{246} + x^{245} + 2x^{244} + 3x^{243} + 4x^{242} + 2x^{241} + x^{240} + x^{236} + 3x^{235} + 4x^{233} + 3x^{232} + x^{231} + 3x^{228} + x^{227} + 3x^{226} + 2x^{225} + 3x^{224} + 3x^{223} + 2x^{222} + 3x^{221} + x^{220} + 4x^{219} + 2x^{217} + 3x^{215} + 4x^{213} + x^{212} + x^{211} + x^{210} + x^{209} + 4x^{207} + x^{206} + 2x^{204} + 3x^{202} + 2x^{201} + 2x^{199} + 3x^{197} + x^{196} + 4x^{195} + x^{194} + 3x^{193} + 2x^{192} + 4x^{191} + x^{190} + 2x^{189} + x^{187} + 2x^{186} + 2x^{184} + 4x^{183} + 3x^{182} + 4x^{180} + 3x^{177} + 4x^{175} + x^{174} + 3x^{172} + 4x^{170} + 4x^{169} + 3x^{167} + x^{166} + 3x^{165} + 3x^{164} + 2x^{163} + 4x^{161} + 4x^{160} + 4x^{159} + 4x^{158} + x^{157} + 4x^{156} + 2x^{155} + x^{154} + 4x^{152} + x^{151} + 3x^{149} + 3x^{147} + 2x^{146} + 2x^{145} + 2x^{144} + x^{143} + 3x^{142} + 4x^{141} + 4x^{140} + 2x^{139} + 4x^{138} + x^{137} + 2x^{134} + 3x^{133} + x^{132} + 3x^{130} + 2x^{129} + 4x^{127} + 2x^{126} + x^{124} + 2x^{123} + 4x^{121} + 2x^{119} + 2x^{118} + 4x^{115} + 3x^{114} + x^{113} + 4x^{111} + 2x^{109} + 4x^{108} + 3x^{106} + 4x^{105} + 4x^{104} + x^{103} + x^{102} + x^{100} + x^{99} + x^{98} + 2x^{97} + x^{96} + x^{95} + x^{94} + x^{93} + 2x^{92} + 4x^{91} +$$

Example of Standard representation VI

$$x^{88}+3x^{87}+4x^{86}+2x^{84}+x^{83}+3x^{82}+2x^{81}+2x^{80}+x^{79}+4x^{78}+x^{77}+x^{76}+x^{75}+x^{74}+3x^{73}+ 2x^{71}+3x^{70}+x^{69}+3x^{68}+x^{66}+2x^{65}+3x^{64}+2x^{62}+x^{60}+x^{59}+x^{58}+3x^{56}+2x^{55}+2x^{54}+ 4x^{53}+2x^{52}+2x^{51}+x^{49}+3x^{48}+2x^{47}+3x^{45}+3x^{44}+4x^{43}+2x^{41}+x^{40}+x^{39}+3x^{38}+3x^{37}+ 2x^{36}+3x^{34}+x^{33}+3x^{32}+2x^{31}+4x^{30}+2x^{29}+3x^{27}+3x^{26}+4x^{25}+3x^{23}+3x^{22}+4x^{20}+ 3x^{19}+2x^{17}+x^{15}+x^{14}+x^{13}+2x^{12}+4x^{10}+2x^{9}+3x^{8}+3x^{7}+2x^{6}+3x^{5}+3x^{3}+2x^{2}+x.$$

Example of Compact representation I

For the same fundamental unit ϵ in the example of standard representation can be written as

$$\epsilon = \mu \cdot \frac{1}{(\beta_1^{2^9} \cdot \beta_2^{2^8} \cdot \beta_3^{2^7} \cdot \beta_4^{2^6} \cdot \beta_5^{2^5} \cdot \beta_6^{2^4} \cdot \beta_7^{2^3} \cdot \beta_8^{2^2} \cdot \beta_9^{2} \cdot \beta_{10}^{1})}$$

where $\mu = 2$, $\beta_1 = 2y$,

$$\beta_2 = \beta_3 = 2y^2 + (2x^3 + 2)y + 2x^5 + 2x^4 + 2,$$

$$\beta_4 = \frac{3x^5+4x^4+3x^3+x^2+4x+2}{x^3+2x+1}y^2 + \frac{3x^8+4x^7+3x^6+4x^5+3x^4+x^2+4x+4}{x^3+2x+1}y + \frac{3x^{10}+2x^9+2x^8+4x^7+4x^5+x^4+2x^3+2x^2+3x+3}{x^3+2x+1},$$

$$\beta_5 = \frac{2x^2+1}{x^5+2x^4+4x^3+3x^2+2x}y^2 + \frac{2x^5+2x^3+x^2+2x+2}{x^5+2x^4+4x^3+3x^2+2x}y, + \frac{2x^7+3x^6+4x^5+2x^4+x^3+3x^2+x+3}{x^5+2x^4+4x^3+3x^2+2x}$$

$$\beta_6 = \frac{2x^4+3x^3+x+1}{x^4+4x^3+4x+2}y^2 + \frac{2x^7+3x^6+3x^4+4x^3+4x+2}{x^4+4x^3+4x+2}y,$$

Example of Compact representation II

$$+\frac{2x^9+3x^7+x^6+2x^5+3x^4+2x^3+x^2+4x+2}{x^4+4x^3+4x+2},$$

$$\beta_7 = \frac{2x^6+2x^5+x^4+4x^3+x^2+3}{x^4+3x^3+3x^2+3x+4}y^2 + \frac{2x^9+2x^8+x^7+x^6+3x^5+x^4+2x^3+x^2+3x}{x^4+3x^3+3x^2+3x+4}y$$
$$+\frac{2x^{11}+4x^{10}+3x^9+3x^6+4x^4+4x^2+2}{x^4+3x^3+3x^2+3x+4},$$

$$\beta_8 = \frac{3x^5+2x^4+2x^3+2x^2+4x+1}{x^6+4x^5+2x^4+3x^3+2x^2+x+4}y^2 + \frac{3x^8+2x^7+2x^6+x^4+2x^3+x+1}{x^6+4x^5+2x^4+3x^3+2x^2+x+4}y$$
$$+\frac{(3x^{10}+4x^8+4x^7+4x^6+x^5+x^4+x^3+3}{x^6+4x^5+2x^4+3x^3+2x^2+x+4},$$

$$\beta_9 = \frac{2x^6+4x^5+x^4+3x^3+3x^2+3x+3}{x^5+3x^4+2x^3+4x^2+x}y^2 + \frac{2x^9+4x^8+x^7+2x^5+4x^4+4x^3+3x^2+4x+1}{x^5+3x^4+2x^3+4x^2+x}y$$
$$+\frac{2x^{11}+x^{10}+4x^8+x^7+3x^6+4x^5+x^4+3x^3+2x+4}{x^5+3x^4+2x^3+4x^2+x},$$

$$\beta_{10} = (4x^5+4x^4+3x^3+4x^2+4)y^2$$
$$+(4x^8+4x^7+3x^6+3x^5+4x^4+2x^3+4x^2+1)y$$
$$+4x^{10}+3x^9+2x^8+2x^7+4x^6+3x^5+3x^4+4x^2.$$