Algorithms for p-adic heights on hyperelliptic curves of arbitrary reduction

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Introduction

Let p be an odd prime number. In the literature, there are several definitions of p-adic height pairings on abelian varieties defined over $\mathbb{Q}$. Some of the definitions were given by Schneider, Mazur–Tate and Nekovář. For Jacobians of curves, there is another definition due to Coleman–Gross.

Algorithms for computing p-adic heights

- allow one to compute p-adic regulators, some of which fit into p-adic versions of BSD conjecture, and
- play a crucial role in carrying out the quadratic Chabauty method to determine integral/rational points on curves of genus at least two.

Goal

Present algorithms to compute

- Coleman–Gross height on hyperelliptic Jacobians, and
- Mazur–Tate height on Jacobian surfaces.

Overview

1. The Coleman–Gross p-adic height pairing
   - Basic definitions
   - The local component at p
   - Computation of Ψ
   - Our algorithm
   - Numerical example: elliptic curve
2. The Mazur–Tate p-adic height function
   - Decomposition into p-adic Néron functions
   - Our algorithm
   - Numerical example: quadratic Chabauty for integral points

§1. The Coleman–Gross p-adic height pairing

Let $C/\mathbb{Q}$ be a nice curve. The Coleman–Gross pairing

$$h^{\mathrm{CG}}\colon \mathrm{Div}^0(C) \times \mathrm{Div}^0(C) \to \mathbb{Q}_p$$

is defined as

$$h^{\mathrm{CG}} = h_p^{\mathrm{CG}} + \sum_{q \neq p} h_q^{\mathrm{CG}}.$$

The local components away from p are described using "arithmetic intersection theory" and their computation is standard these days, but

$$h_p^{\mathrm{CG}}(D_1, D_2) := {}^{\mathrm{Vol}}\!\!\int_{D_2} \omega_{D_1}$$

where

- ω_{D_1} is a "canonical" differential form attached to D_1, and
- ${}^{\mathrm{Vol}}\!\!\int$ is the Vologodsky integration.

From now on, everything is local; so set $X = C \otimes \mathbb{Q}_p$.

§1. Vologodsky integration

Let ω be a meromorphic 1-form on X, let $P, Q \in X(\mathbb{Q}_p)$. To this data, Vologodsky associated an integral

$$\prescript{\mathrm{Vol}}{}{\int_P^Q} \omega \in \mathbb{Q}_p$$

that satisfies the desirable properties. It's called the **Vologodsky integral**.

Remarks:

1. When X has good reduction,
   Vologodsky integral $=$ Coleman integral.
2. When ω is holomorphic,
   Vologodsky integral $=$ abelian integral.
3. There are practical algorithms to compute these:
   - Balakrishnan–Tuitman: Coleman integrals on smooth curves,
   - Katz–K: Vologodsky integrals on hyperelliptic curves,
   - Katz–K (in progress): Vologodsky integrals on "schön" curves.

§1. The logarithm map Ψ

I. A meromorphic 1-form ω on X is of the **third kind** if ω has at most simple poles with integer residues.

II. Let $H^1_{\mathrm{dR}}(X)$ denote the first de Rham cohomology group of X. Let

$$\cup \colon H^1_{\mathrm{dR}}(X) \times H^1_{\mathrm{dR}}(X) \to \mathbb{Q}_p$$

denote the algebraic cup product pairing on $H^1_{\mathrm{dR}}(X)$.

III. Let Ψ be the "logarithm for the universal vectorial extension" of the Jacobian of X. It is a homomorphism

$$\{\text{third kind differentials}\} \to H^1_{\mathrm{dR}}(X)$$

which is the identity on holomorphic differentials.

§1. The local component at p

Let $W = W_p$ be a subspace of $H^1_{\mathrm{dR}}(X)$ that is complementary to the space of holomorphic forms:

$$H^1_{\mathrm{dR}}(X) = H^0(X, \Omega^1_X) \oplus W.$$

There exists a unique form ω_{D_1} of the third kind satisfying

$$\mathrm{Res}(\omega_{D_1}) = D_1, \quad \Psi(\omega_{D_1}) \in W.$$

Then

$$h_p^{\mathrm{CG}}(D_1, D_2) := {}^{\mathrm{Vol}}\!\!\int_{D_2} \omega_{D_1}.$$

Remarks:

- When X has semistable ordinary reduction, there is a canonical choice for W: the unit root subspace for the action of the Frobenius endom.
- An algorithm to compute h_p^{CG} when X is hyperelliptic with good reduction was provided by Balakrishnan–Besser.

§1. Computation of Ψ

A crucial step in computing $h_p^{\mathrm{CG}}(D_1, D_2)$ is the construction of the form ω_{D_1}. This requires the explicit computation of the map Ψ, but its original definition is not suitable for computations.

Proposition (Besser): The logarithm map Ψ can be expressed in terms of the cup product and "global symbol".

Global symbol is defined in terms of Vologodsky integration, so

$$\text{Computing } \Psi \quad \approx \quad \begin{array}{c}\text{computing Vol.}\\ \text{integrals}\end{array} + \begin{array}{c}\text{computing cup}\\ \text{products}\end{array}$$

§1. Computing CG heights on hyperelliptic curves

Let $C/\mathbb{Q}$ be a genus-g hyperelliptic curve with affine model

$$y^2 = f(x), \quad f(x) \in \mathbb{Z}[x] \text{ is monic.}$$

For simplicity, we assume that $\deg(f(x))$ is odd.

The height pairing is bilinear. Therefore we may assume that

$$D_1 = (P) - (Q), \quad D_2 = (R) - (S).$$

We also assume that P, Q, R, S are pairwise distinct.

Set $X = C \otimes \mathbb{Q}_p$. Let $W = W_p$ be a subspace of $H^1_{\mathrm{dR}}(X)$ such that

$$H^1_{\mathrm{dR}}(X) = H^0(X, \Omega^1_X) \oplus W.$$

Step 1. *Pick an* ω *of the third kind with residue divisor* $D_1 = (P) - (Q)$:

$$\omega := \begin{cases} \left(\frac{y + y(P)}{x - x(P)} - \frac{y + y(Q)}{x - x(Q)}\right)\frac{dx}{2y} & \text{if } P \text{ and } Q \text{ are finite;} \\ \frac{y + y(P)}{x - x(P)}\frac{dx}{2y} & \text{if } P \text{ is finite, } Q \text{ is infinite.} \end{cases}$$

Step 2. *Determine the holomorphic form* η *such that* $\Psi(\omega - \eta)$ *lies in* W*; then* $\omega_{D_1} = \omega - \eta$:

I. Compute $\Psi(\omega)$ as an element of $H^1_{\mathrm{dR}}(X)$.

II. Compute $\Psi(\omega)$ as an element of $H^0(X, \Omega^1_X) \oplus W$.

III. Compare them and get η.

§1. Computing CG heights on hyperelliptic curves

Step 3. *Compute the Vologodsky integral* ${}^{\mathrm{Vol}}\!\int_{D_2} \omega_{D_1} = {}^{\mathrm{Vol}}\!\int_S^R (\omega - \eta)$:

I. Cover X by $\{U_i\}_i$ such that each U_i can be embedded into X_i, which is

- either a rational curve,
- or a hyperelliptic curve of good reduction.

II. Express Vologodsky integrals on X as explicit linear combinations of Vologodsky integrals on X_i.

III. In order to compute Vologodsky integrals on X_i,

- parametrize X_i if it is rational,
- make use of the Coleman integration algorithms if X_i is hyperelliptic.

Remark: We can define h_p^{CG} for divisors with common support, but, for instance, $h_p^{\mathrm{CG}}(P - \infty, P - \infty)$ becomes a double Vologodsky integral and we do not know (yet) how to deal with them...

§1. Computing a canonical complementary subspace

All complementary subspaces are equal,
but some are more equal than others.

We now assume that

- X has semistable ordinary reduction, and
- X has genus $g = 2$.

Extending a construction by Blakestad, Bianchi constructed a very "canonical" complementary subspace W^C of $H^1_{\mathrm{dR}}(X)$.

Proposition (Bianchi): If X has good ordinary reduction, then

$$W^C = \text{the unit root subspace.}$$

Algorithm (Bianchi–K–Müller)

We can compute Blakestad's complementary subspace W^C.

§1. Numerical example: elliptic curve

The following curve has split multiplicative reduction at $p = 43$:

$$C\colon y^2 = x^3 - 1351755x + 555015942, \quad P := \left(\tfrac{330483}{361}, \tfrac{63148032}{6859}\right) \in C(\mathbb{Q}).$$

Using `SageMath`, the "canonical" Mazur–Tate height of P is

$$19 \cdot 43 + 7 \cdot 43^2 + 8 \cdot 43^3 + 2 \cdot 43^4 + O(43^5). \tag{1}$$

Canonical MT height = CG height wrt the unit root subspace

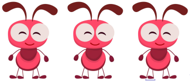
For $Q = (2523, 114912)$, $R = (219, 16416) \in C(\mathbb{Q})$, let

$$D_Q := (Q) - (-Q), \quad D_R := (R) - (-R) \quad \Longrightarrow \quad P = D_Q = D_R.$$

The Coleman–Gross height $h^{\mathrm{CG}}(D_Q, D_R)$ wrt the unit root subspace is

$$\mathbf{9 \cdot \log_{43}(2)} + 29 \cdot 43 + 28 \cdot 43^2 + 10 \cdot 43^3 + 39 \cdot 43^4 + O(43^5). \tag{2}$$

Luckily, (1) and (2) are equal...



§2. The Mazur–Tate p-adic height function

Let $C/\mathbb{Q}$ be a nice curve, and let J be its Jacobian variety. The Mazur–Tate height function

$$h^{\mathrm{MT}} \colon J(\mathbb{Q}) \to \mathbb{Q}_p$$

is defined using "biextensions".

Theorem (Bianchi, Bianchi–K.–Müller)

For each q, there exists a function $\lambda_q \colon J(\mathbb{Q}_q) \to \mathbb{Q}_p$ such that h^{MT} can be decomposed into a sum of λ_q's.

*We call λ_q the p-**adic Néron function** at q. It is*

- *the real-valued Néron function (up to a constant) for $q \neq p$;*
- *defined using Besser's "p-adic log" function for $q = p$.*

This is a p-adic analogue of a classical result in Diophantine geometry: the Néron–Tate height can be decomposed into a sum of Néron functions.

§2. Computing MT heights on Jacobian surfaces

Assume $C/\mathbb{Q}$ is of genus-2 with affine model

$$y^2 = f(x), \quad f(x) \in \mathbb{Z}[x] \text{ is quintic.}$$

Then extending a construction by Blakestad, Bianchi defined a 2-dimensional p-adic sigma function σ_p: it is a certain solution of a differential equation inside the formal group of J.

Algorithm (Bianchi, Bianchi–K.–Müller)

The p-adic Néron function λ_p is essentially the p-adic log of σ_p. Therefore, we can compute the Mazur–Tate height function h^{MT} on J.

This is a genus-2 analogue of a classical result: global p-adic heights on elliptic curves can be expressed in terms of 1-dimensional p-adic sigma functions (this goes back to Bernardi, Mazur–Tate, Mazur–Stein–Tate...).

§2. Num. example: quadratic Chabauty for integral points

Method: It uses properties of local and global heights to produce a locally analytic function

$$\rho\colon C(\mathbb{Z}_p) \to \mathbb{Q}_p$$

such that $\rho(C(\mathbb{Z})) \subset \Gamma$ for an effectively computable finite set Γ. The method requires the computation of local heights of the form $h_q(P - \infty, P - \infty)$. With our p-adic Néron function approach, this is not a problem at all:

Example: The following curve has bad reduction at $p = 5$:

$$C\colon y^2 = x^5 + x^3 - 2x + 1.$$

Applying the quadratic Chabauty method for $p = 5$, we get

$$C(\mathbb{Z}) = \{(0, \pm 1), (1, \pm 1), (-1, \pm 1)\}.$$

Remark: This is, to the best of our knowledge, the first worked quadratic Chabauty example in the literature at a prime of bad reduction...

Thank you! Dank u wel! Teşekkür ederim!

- *p-adic heights on curves* - Coleman–Gross
- *Hodge structure on the fundamental group and its application to p-adic integration* - Vologodsky
- *p-adic heights and Vologodsky integration* - Besser
- *p-adic Arakelov theory* - Besser
- *Computing local p-adic height pairings on hyperelliptic curves* - Balakrishnan–Besser
- *Coleman-Gross height pairings and the p-adic sigma function* - Balakrishnan–Besser
- *Quadratic Chabauty: p-adic height pairings and integral points on hyperelliptic curves* - Balakrishnan–Besser–Müller
- *Explicit Coleman integration for curves* - Balakrishnan–Tuitman
- *On generalizations of p-adic Weierstrass sigma and zeta functions* - Blakestad

- *p-adic integration on bad reduction hyperelliptic curves* - Katz–Kaya
- *Explicit Vologodsky integration for hyperelliptic curves* - Kaya
- *p-adic sigma functions and heights on Jacobians of genus 2 curves* - Bianchi
- *Coleman–Gross heights and p-adic Néron Functions on Jacobians of genus 2 curves* - Bianchi–Kaya–Müller
- *Algorithms for p-adic heights on hyperelliptic curves of arbitrary reduction* - Bianchi–Kaya–Müller