Isogeny interpolation for elliptic curves, and applications

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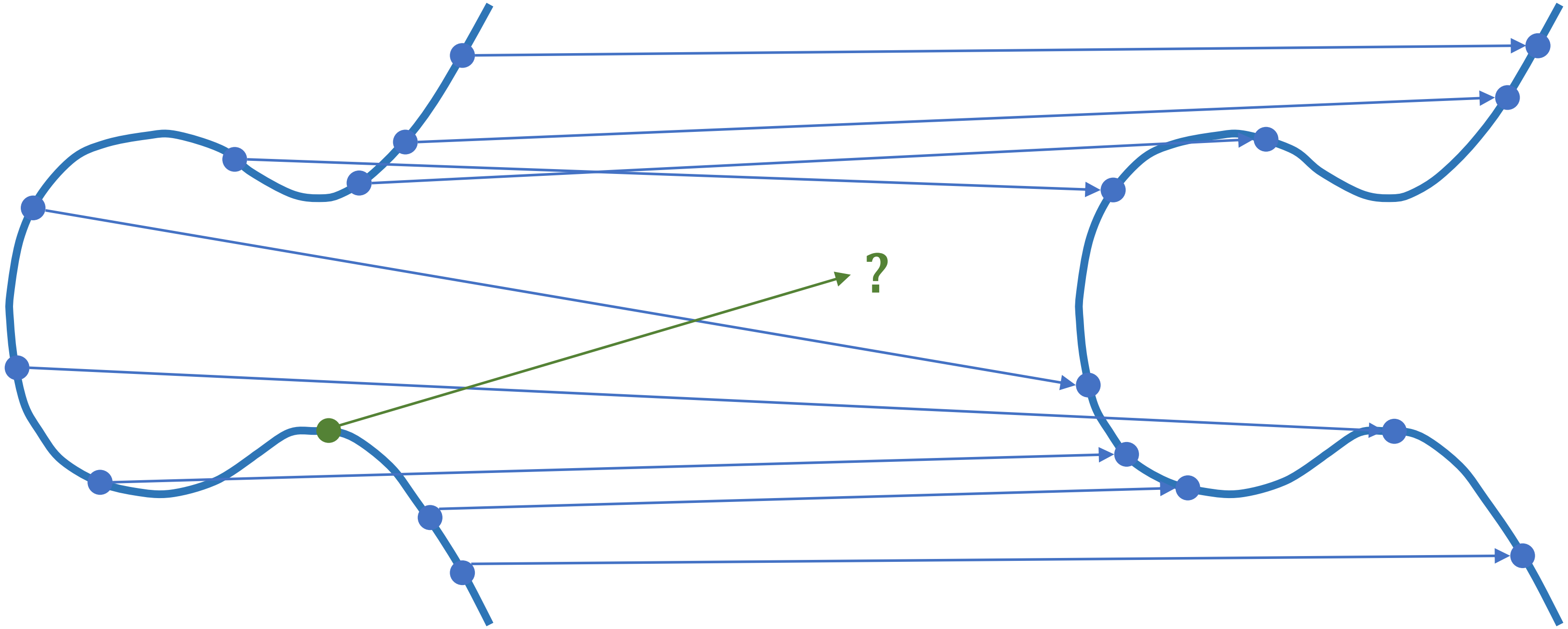

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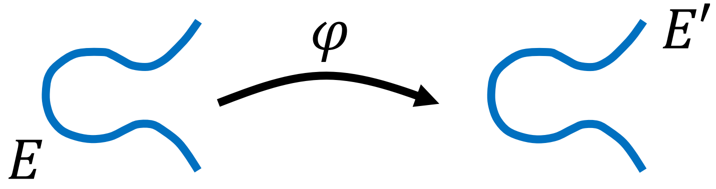
1. Problem statement and main result

Definition

A **homomorphism** between two elliptic curves E and E' over a field k is a morphism $\varphi\colon E \to E'$ such that $\varphi(\infty) = \infty'$.

An **isogeny** is a non-constant homomorphism.



<u>Example:</u> let $E\colon y^2 = x^3 + 1$ and $E'\colon y^2 = x^3 - 27$, then

$$\varphi\colon E \to E'\colon \infty \mapsto \infty, (x,y) \mapsto \begin{cases} \left(\dfrac{x^3+4}{x^2}, y\dfrac{x^3-8}{x^3}\right) & \text{if } (x,y) \neq (0, \pm 1), \\ \infty \text{ if not} & \end{cases}$$

is an isogeny of degree 3.

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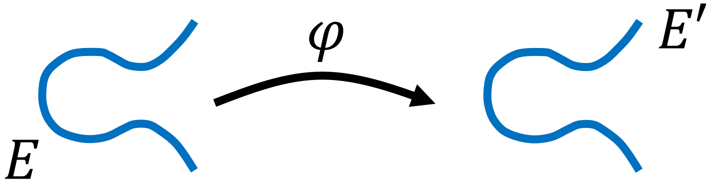

Quick facts:

- on $\bar{k}$-points, isogenies are **surjective group homomorphisms** with finite kernel,
- $\#\ker\varphi \leq \deg\varphi$, where equality holds if and only if φ is separable.

Cauchy—Schwarz inequality

For any pair of isogenies $\varphi_1\colon E \to E'$ and $\varphi_2\colon E \to E'$ we have:

$$|\deg(\varphi_1 - \varphi_2) - \deg\varphi_1 - \deg\varphi_2| \leq 2\sqrt{\deg\varphi_1 \deg\varphi_2}.$$

E.g., **Hasse's theorem** for $k = \mathbf{F}_q$: $\left|\#E\left(\mathbf{F}_q\right) - q - 1\right| = \left|\deg\left(\mathrm{Frob}_q - \mathrm{id}\right) - q - 1\right| \leq 2\sqrt{q}$.

1. Problem statement and main result

Corollary [JU18]

A degree-d isogeny $\varphi: E \to E'$ is **uniquely determined** by the images of any $4d+1$ points.

<u>Proof:</u>
- Let $\varphi_1 \neq \varphi_2$ be degree-d isogenies coinciding on $\geq 4d+1$ points.
- Then $\deg(\varphi_1 - \varphi_2) \geq \#\ker(\varphi_1 - \varphi_2) \geq 4d+1$, but by Cauchy—Schwarz:

$$|\deg(\varphi_1 - \varphi_2) - d - d| \leq 2\sqrt{d \cdot d} \quad \Rightarrow \quad \deg(\varphi_1 - \varphi_2) \leq 4d \;\; ↯$$

Bound is **sharp**: φ and $-\varphi$ agree on $[2]^{-1}(\ker \varphi)$.

Isogeny interpolation problem

- Let $\varphi: E \to E'$ be an unknown isogeny of (known) degree d.
- Let us be given a set $\{P_1, \dots, P_r\} \subset E$ generating a group of size at least $4d+1$, along with the image points $\varphi(P_i) \in E'$ for $i = 1, \dots, r$.
- Given any $Q \in E$, compute $\varphi(Q)$.

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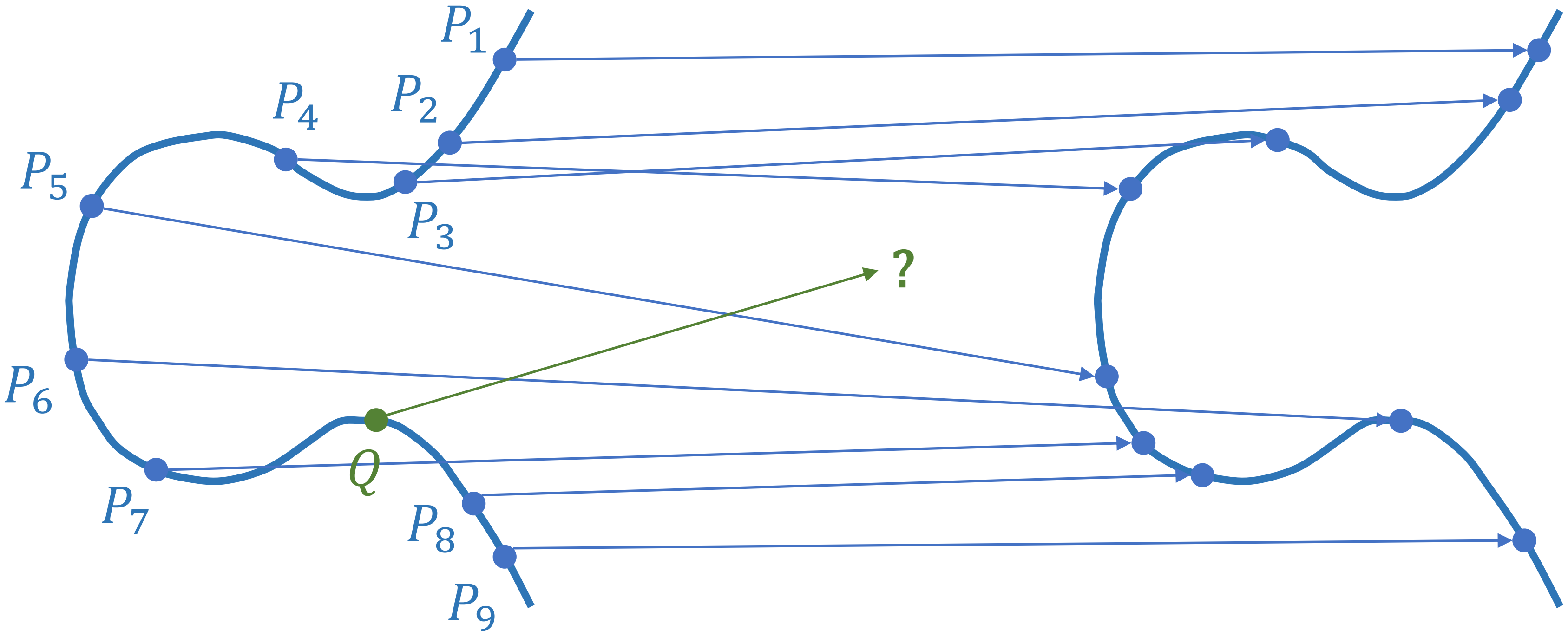

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Theorem [CDM+24]

Assume $k = \mathbf{F}_q$, write $G = \langle P_1, \ldots, P_r \rangle$, and assume $\gcd(\#G, q) = 1$. There is an algorithm for the isogeny interpolation problem, whose running time is polynomial in:

- the length of the input,
- the maximum of ℓ and the degree of the field of definition of $E\left[\ell^{\lfloor e/2 \rfloor}\right]$, over all prime powers ℓ^e dividing $\#G$.

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  over all prime powers ℓ^e dividing $\#G$.

Remarks:

- polynomial time, but concrete **runtime varies largely with parameters**,
- supersingular: conditions on $\gcd(\#G, q)$ and $E\left[\ell^{\lfloor e/2 \rfloor}\right]$ are void,
- ordinary: condition on $\gcd(\#G, q)$ likely removable via Dieudonné modules,
- much generalizes to p.p. abelian varieties of dim $g \geq 1$, or to arbitrary fields supporting efficient arithmetic, but full extent not clear yet.

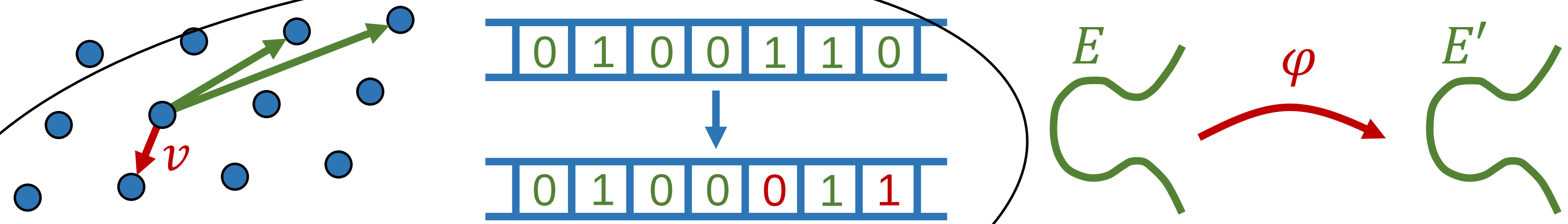
2. Context: post-quantum cryptography standardization

1994: “Polynomial-time algorithms for **prime factorization** and **discrete logarithms** on a quantum computer” by P. Shor [Sho94]

gradually growing concern

2017: NIST initiates “standardization effort” for post-quantum key exchange and signatures

Main contending **hard mathematical problems**:



$$\begin{cases} f_1(s_1, \dots, s_n) = 0 \\ \vdots \\ f_m(s_1, \dots, s_n) = 0 \end{cases}$$

finding short vectors in lattices

decoding for random linear codes

finding isogenies between elliptic curves

solving non-linear systems of equations

“noisy linear algebra”

2. Context: post-quantum cryptography standardization

2020: Preliminary NIST standards:

- **LMS** (stateful signatures)
- **XMSS** (stateful signatures)

2022: First main NIST standards:

- **Kyber** (key encapsulation)
- **Dilithium** (signatures)
- **Falcon** (signatures)
- **SPHINCS+** (signatures)

few months earlier [Beu22]

$$\begin{cases} f_1(s_1, \dots, s_n) = 0 \\ \vdots \\ f_m(s_1, \dots, s_n) = 0 \end{cases}$$

~~**Rainbow** (signatures)~~

now broken [CD23,MMP+23,Rob23]

Moved to extra round of scrutiny:

- **BIKE** (key encapsulation)
- **McEliece** (key encapsulation)
- **HQC** (key encapsulation)
- ~~**SIKE** (key encapsulation)~~

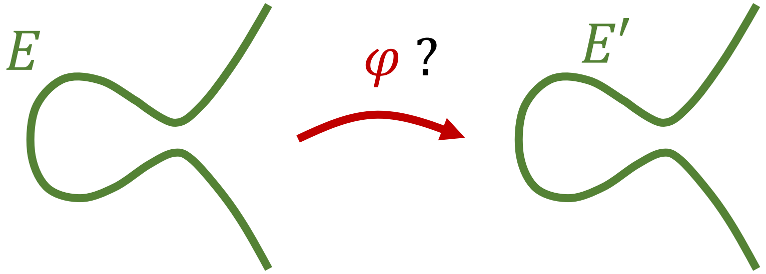
2023: New NIST call:

verification. NIST is open to receiving additional submissions based on structured lattices, but is intent on diversifying the post-quantum signature standards. As such

3. Isogeny-based cryptography 1.x

The isogeny-finding problem:

Input: two isogenous elliptic curves E, E' over $\mathbf{F}_q$



Output: an isogeny $\varphi: E \to E'$

easily tested via point counting, in view of Tate's isogeny theorem [Tat66]

e.g., as a composition of small-degree isogenies

SEE LATER

3. Isogeny-based cryptography 1.x

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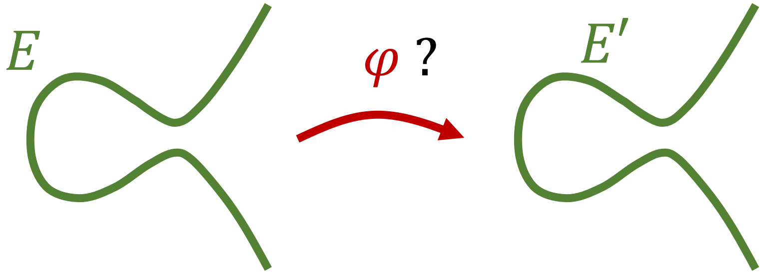

Output: an isogeny $\varphi: E \to E'$

Best attacks in general:[1, 2] $\tilde{O}(q^{1/4})$ classical and $\tilde{O}(q^{1/8})$ quantum [BJS14].

Main selling point: low bandwidth requirements (as in classical elliptic-curve cryptography)

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[1] Large classes of elliptic curves admit more efficient attacks.

[2] For exceptional classes of ordinary elliptic curves, the best-known complexity worsens to $\tilde{O}(q^{2/5})$; see talk Steven Galbraith.

3. Isogeny-based cryptography 1.x

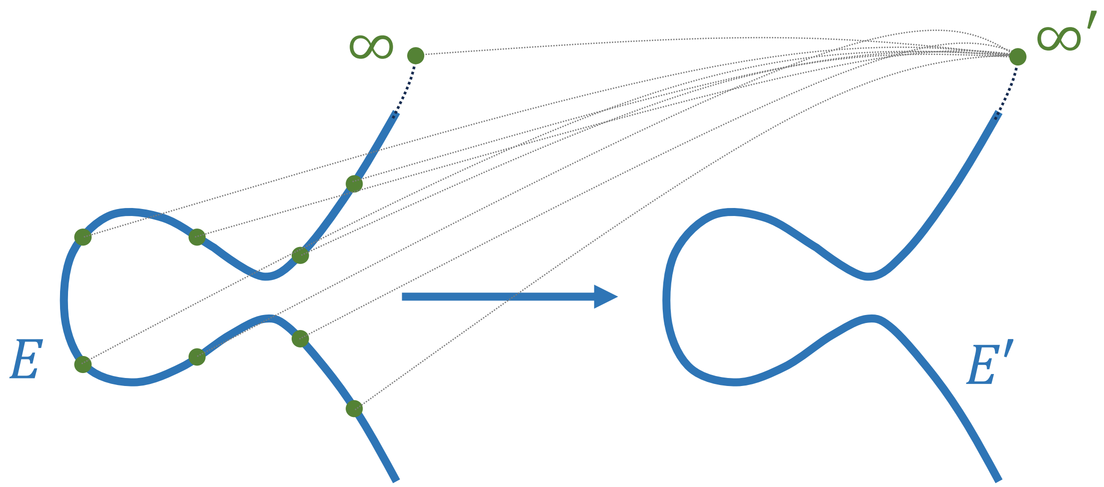
Flagship protocol until 2022: Supersingular Isogeny Key Encapsulation [JD11]

- Another quick fact: for any finite subgroup $G \subset E$ there exists a separable isogeny

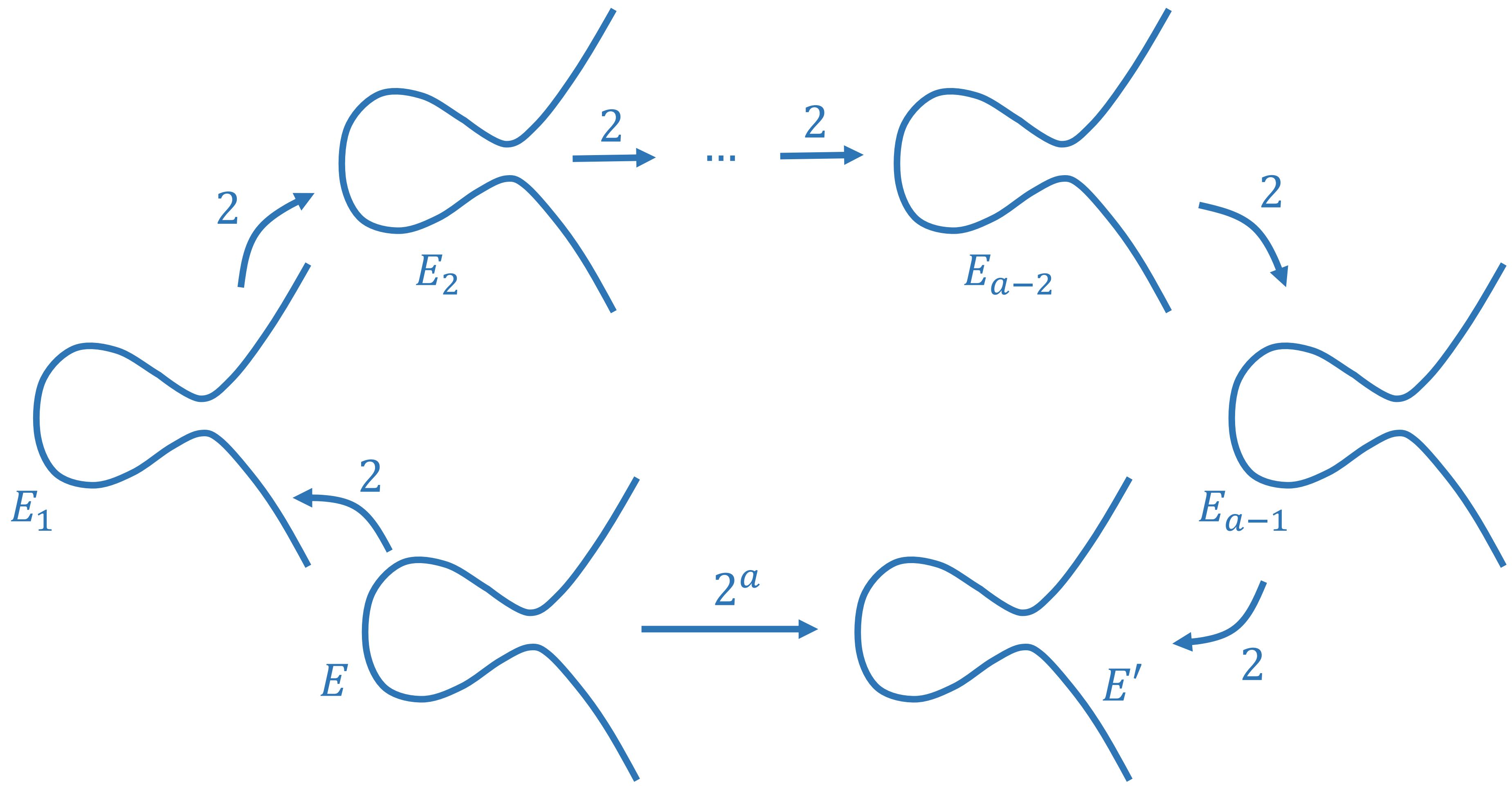
$$\varphi: E \to E' \text{ with } G = \ker(\varphi)$$

and this isogeny is unique up to composition (on the right) with an isomorphism.

makes sense to write $E' = E/G$



- Vélu's formulas [Vél73] compute φ and E', but only efficient when $\#G$ is smooth.



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3. Isogeny-based cryptography 1.x

SIKE's high-level idea: Alice and Bob choose secret subgroups $A \subset E[2^a]$, $B \subset E[3^b]$

$$
\begin{tikzcd}
E \arrow[r, "\varphi_A"] \arrow[d, "\varphi_B"'] & E_A = E/A \arrow[d] \\
E_B = E/B \arrow[dr] & E_{AB} = E_A/\varphi_A(B) \\
 & E/(A+B) \arrow[u, phantom, "\cong"] \\
 & E_{BA} = E_B/\varphi_B(A) \arrow[u, phantom, "\cong"]
\end{tikzcd}
$$

Constructive problem:

allow Bob to determine $\varphi_A(B)$
without revealing φ_A

... and likewise
for Alice

3. Isogeny-based cryptography 1.x

Solution [JD11]: Alice and Bob choose public bases $P_A, Q_A \in E[2^a]$, $P_B, Q_B \in E[3^b]$

$$\begin{tikzcd}
E \arrow[rr, "\varphi_A", "A = \langle P_A + \alpha Q_A \rangle"'] \arrow[d, "\varphi_B"', "B = \langle P_B + \beta Q_B \rangle"] & & E_A = E/A \arrow[d] \\
E_B = E/B \arrow[rr] & & E_{BA} \cong E_{AB}
\end{tikzcd}$$

Alice reveals
$\varphi_A(P_B), \varphi_A(Q_B)$

allows Bob to compute
$\varphi_A(B) = \langle \varphi_A(P_B) + \beta \varphi_A(Q_B) \rangle$

Bob reveals
$\varphi_B(P_A), \varphi_B(Q_A)$ $\longrightarrow$ allows Alice to compute $\varphi_B(A) = \langle \varphi_B(P_A) + \alpha \varphi_B(Q_A) \rangle$

3. Isogeny-based cryptography 1.x

Solution [JD11]: Alice and Bob choose public bases $P_A, Q_A \in E[2^a]$, $P_B, Q_B \in E[3^b]$

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Alice reveals
$\varphi_A(P_B), \varphi_A(Q_B)$

Bob reveals
$\varphi_B(P_A), \varphi_B(Q_A)$

Dramatically weakens the system!

- compute $\varphi_A(P_A), \varphi_A(Q_A)$ using isogeny interpolation
- find α as an (easy) discrete log.

4. Isogeny interpolation: balanced case

"The number of curves of genus two with elliptic differentials" by E. Kani [Kan97]

Lemma. *Consider a commuting diagram of isogenies:*

$$\begin{tikzcd} E_1 \arrow[r, "\beta"] \arrow[d, "\alpha"'] & E_3 \arrow[d, "\gamma"] \\ E_2 \arrow[r, "\delta"'] & E_4 \end{tikzcd}$$

same degree

same degree

4. Isogeny interpolation: balanced case

"The number of curves of genus two with elliptic differentials" by E. Kani [Kan97]

Lemma. *Consider a commuting diagram of isogenies:*

"isogeny diamonds"

$$\begin{tikzcd} E_1 \arrow[r, "\beta"] \arrow[d, "\alpha"'] & E_3 \arrow[d, "\gamma"] \\ E_2 \arrow[r, "\delta"'] & E_4 \end{tikzcd}$$

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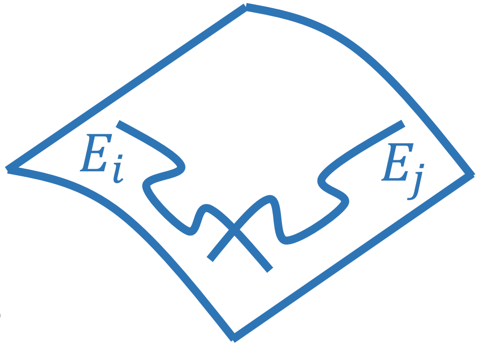


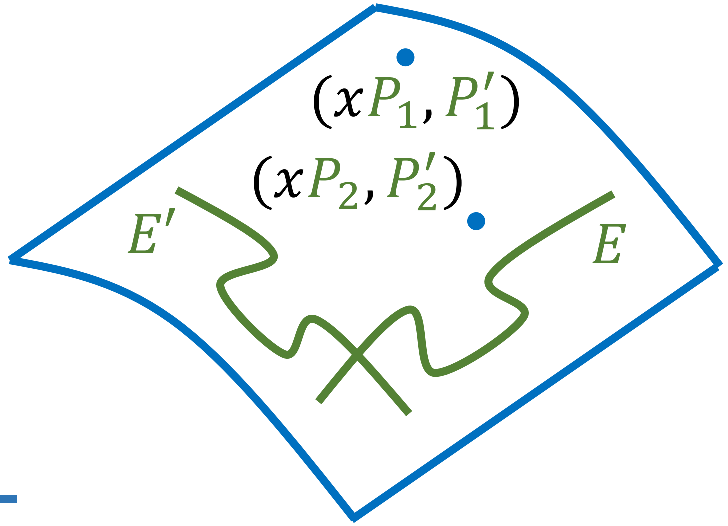
with $\deg\alpha = \deg\gamma$ *and* $\deg\beta = \deg\delta$ *coprime. Then the map*

$$\Phi : E_2 \times E_3 \xrightarrow{\begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ -\delta & \gamma \end{pmatrix}} E_1 \times E_4$$

is a $(\deg\alpha + \deg\beta, \deg\alpha + \deg\beta)$*-isogeny of p.p. abelian surfaces with kernel*

$$\{ (\alpha(P), \beta(P)) \mid P \in E_1[\deg\alpha + \deg\beta] \}.$$

4. Isogeny interpolation: balanced case



Assume $G = E[N] = \langle P_1, P_2 \rangle$ with $N^2 \geq 4d + 1$.

$$\begin{array}{ccc} E & \xrightarrow{\quad\quad\varphi\quad\quad} & E' \\ P_1, P_2 & & P_1' = \varphi(P_1), P_2' = \varphi(P_2) \end{array}$$

Special first case: $\gcd(N, d) = 1$ and $N > d$
$N - d = x^2$ is square

Consider the isogeny diamond

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "{[x]}"'] & E' \arrow[d, "{[x]}"] \\ E \arrow[r, "\varphi"'] & E' \end{tikzcd}$$

Note: $\deg \varphi + \deg[x] = d + x^2 = N$.

Kani's lemma:

$$\Phi : E \times E' \xrightarrow{\begin{pmatrix} [x] & \hat{\varphi} \\ -\varphi & [x] \end{pmatrix}} E \times E'$$

is an (N, N)-isogeny with kernel

$$\{ (xP, \varphi(P)) \mid P \in E[N] \}.$$

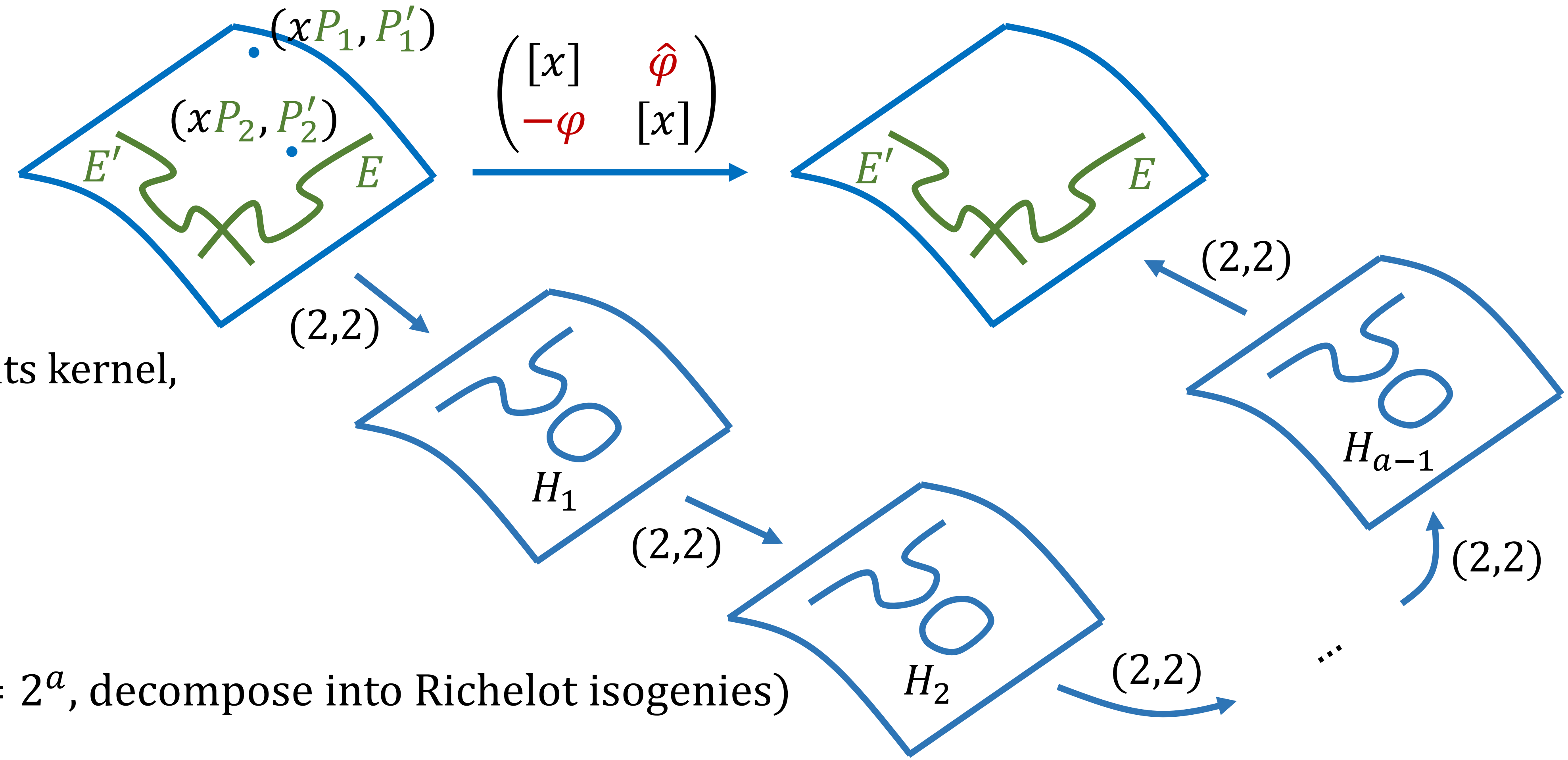
completely known!

4. Isogeny interpolation: balanced case

Algorithm (requires N **smooth**):

- using higher-dimensional analogs of Vélu's formulae, compute the (N, N)-isogeny

Φ :



from its kernel,

(e.g., if $N = 2^a$, decompose into Richelot isogenies)

4. Isogeny interpolation: balanced case

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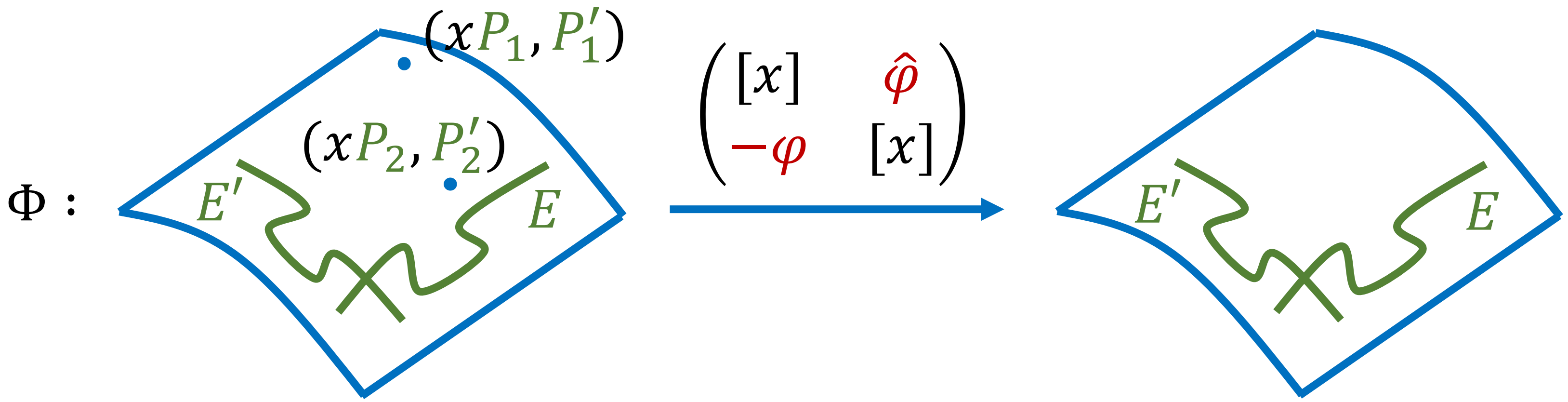
- using higher-dimensional analogs of Vélu's formulae, compute the (N, N)-isogeny



from its kernel,

- compute $-\Phi(Q, \infty) = (-xQ, \varphi(Q))$,
- extract $\varphi(Q)$ as the second component.

4. Isogeny interpolation: balanced case

Assume $G = E[N] = \langle P_1, P_2 \rangle$ with $N^2 \geq 4d + 1$.

$$\begin{array}{ccc} E & \xrightarrow{\qquad\qquad\varphi\qquad\qquad} & E' \\ P_1, P_2 & & P_1' = \varphi(P_1), P_2' = \varphi(P_2) \end{array}$$

Next case: $\gcd(N, d) = 1$ and $N > d$
$N - d = x_1^2 + x_2^2$ is sum of two squares

Same, but use

$$\Phi : E^2 \times E'^2 \xrightarrow{\begin{pmatrix} [x_1] & [x_2] & \hat{\varphi} & 0 \\ [-x_2] & [x_1] & 0 & \hat{\varphi} \\ -\varphi & 0 & [x_1] & [-x_2] \\ 0 & -\varphi & [x_2] & [x_1] \end{pmatrix}} E^2 \times E'^2$$

(higher-dimensional variant of Kani's lemma).

4. Isogeny interpolation: balanced case

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Next case: $\gcd(N, d) = 1$ and $N > d$

$N - d = x_1^2 + x_2^2 + x_3^2 + x_4^2$ via Lagrange's four-square theorem

Now work on $E^4 \times E'^4$ and use (**Zarhin's trick**)

$$\begin{pmatrix} [x_1] & [-x_2] & [-x_3] & [-x_4] & \hat{\varphi} & 0 & 0 & 0 \\ [x_2] & [x_1] & [x_4] & [-x_3] & 0 & \hat{\varphi} & 0 & 0 \\ [x_3] & [-x_4] & [x_1] & [x_2] & 0 & 0 & \hat{\varphi} & 0 \\ [x_4] & [x_3] & [-x_2] & [x_1] & 0 & 0 & 0 & \hat{\varphi} \\ -\varphi & 0 & 0 & 0 & [x_1] & [x_2] & [x_3] & [x_4] \\ 0 & -\varphi & 0 & 0 & [-x_2] & [x_1] & [-x_4] & [x_3] \\ 0 & 0 & -\varphi & 0 & [-x_3] & [x_4] & [x_1] & [-x_2] \\ 0 & 0 & 0 & -\varphi & [-x_4] & [-x_3] & [x_2] & [x_1] \end{pmatrix}$$

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Near-general case: $\gcd(N, d) = 1$
$N - d = x_1^2 + \cdots + x_r^2$ is sum of $r = 1,2,4$ squares

Approach: proceed **as if we would know** the images of $\frac{1}{N}P_1, \frac{1}{N}P_2 \in E[N^2]$.

$$\begin{array}{ccc} A & \xrightarrow{\quad\Phi?\quad} & A \\ \| & & \\ E^r \times E'^r & & \end{array}$$

we no longer know $\ker \Phi$...

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$$\begin{array}{ccccc} A & \xrightarrow{\quad\Phi_1\quad} & X & \xleftarrow{\quad\widehat{\Phi}_2\quad} & A \\ \parallel & & & & \\ E^r \times E'^r & & & & \end{array}$$

we also know $N(\ker \widehat{\Phi})$

but we do know $N(\ker \Phi)$!

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$$\begin{array}{ccccc} A & \xrightarrow{\quad\Phi_1\quad} & X & \xleftarrow{\quad\widehat{\Phi}_2\quad} & A \\ \parallel & & & & \\ E^r \times E'^r & & & & \end{array}$$

so we recover Φ as $\widehat{\widehat{\Phi}}_2 \circ \Phi_1$

(gluing in the middle is subtle)

5. Isogeny interpolation: glimpse at the general case

E.g., assume $G = \langle P_1 \rangle$ cyclic of order $N \geq 4d + 1$.

$$E \xrightarrow{\varphi} E'$$

$$P_1 \qquad\qquad P_1' = \varphi(P_1)$$

$$P_2 \qquad\qquad \varphi(P_2) = \mu P_2' + \lambda P_1'$$

Approach:

- extend to bases $P_1, P_2 \in E[N]$ and $P_1', P_2' \in E'[N]$,

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Approach:

- extend to bases $P_1, P_2 \in E[N]$ and $P_1', P_2' \in E'[N]$,
- use identity $e_N\big(\varphi(P_1), \varphi(P_2)\big) = e_N\big(P_1', \varphi(P_2)\big) = e_N(P_1, P_2)^d$ to determine μ,

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$$
\begin{array}{ccccc}
E & \xrightarrow{\quad\varphi\quad} & E' & \xrightarrow{\quad\psi\quad} & E'/\langle P_1' \rangle \\
P_1 & & P_1' = \varphi(P_1) & & \infty \\
P_2 & & \varphi(P_2) = \mu P_2' + \lambda P_1' & & \mu\psi(P_2')
\end{array}
$$

Approach:

- extend to bases $P_1, P_2 \in E[N]$ and $P_1', P_2' \in E'[N]$,
- use identity $e_N\big(\varphi(P_1), \varphi(P_2)\big) = e_N\big(P_1', \varphi(P_2)\big) = e_N(P_1, P_2)^d$ to determine μ,
- compose with $\psi: E' \to E'/\langle P_1' \rangle$,
- apply (slight generalization of) previous algorithm to $G = E[N]$ and $\psi \circ \varphi$.

note: $N^2 \geq 4Nd + N \geq 4Nd + 1 = 4\deg(\psi \circ \varphi) + 1$

6. Isogeny representations

What does it mean to **represent** / **output** a degree-d isogeny $\varphi: E \to E'$?

- As a **rational map** ?

$$\text{E.g.,} \quad \varphi : (x, y) \mapsto \left(\frac{x^3 + x^2 + x + 2}{(x-5)^2}, y\frac{x^3 - 4x^2 + 2}{(x-5)^3} \right)$$

Object of size $O((\log q)\, d)$.

Feasible **only if d is smooth** $\longrightarrow$ write φ as a composition of small-degree isogenies.

pre-2022: default understanding of isogeny representation

6. Isogeny representations

What does it mean to represent / output a degree-d isogeny $\varphi: E \to E'$?

- Via its kernel G ?

  If the points in G are defined over $\mathbf{F}_{q^f}$: object of size $O((\log q) f)$.

  Requires conversion to be useful (e.g., to rational map via Vélu, needs smoothness).

- For certain isogenies: via its kernel ideal I_φ (via Deuring correspondence) ?

  Requires sufficient knowledge of the endomorphism ring.

  Requires conversion to be useful; ideal can often be smoothened, e.g., via [KLP+14].

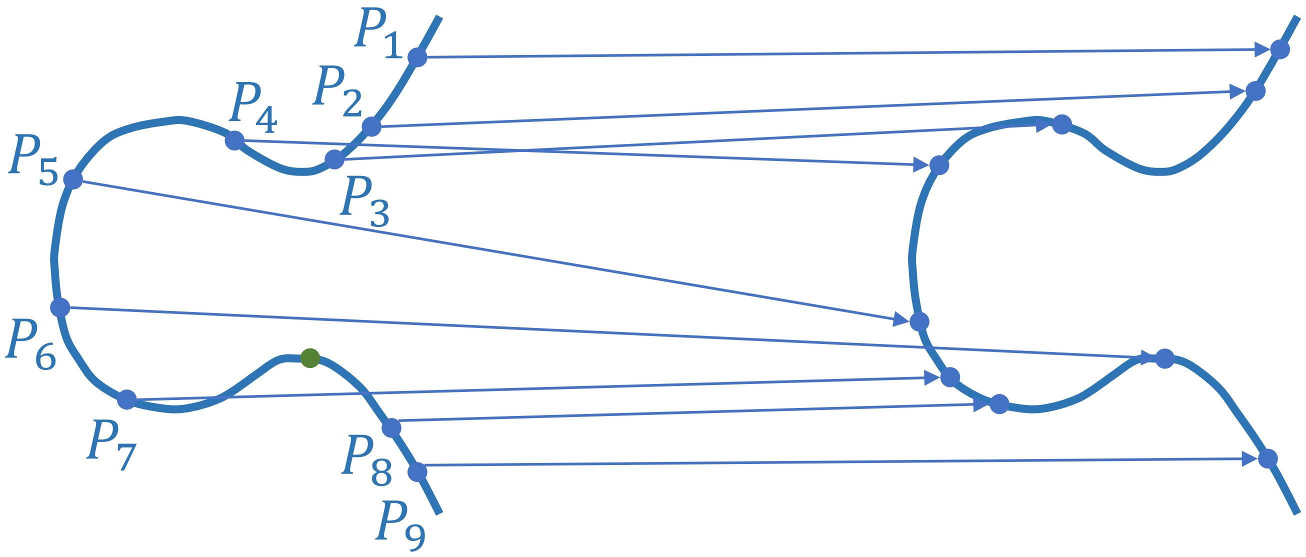
$$\begin{tikzcd} E \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\psi"'] & E' \end{tikzcd} \qquad \varphi = \frac{1}{\deg \psi} \cdot \psi \circ (\hat{\psi} \circ \varphi)$$

SEE LATER

6. Isogeny representations

What does it mean to **represent** / **output** a degree-d isogeny $\varphi : E \to E'$?

➢ Via **interpolation data** !



Two caveats:

- interpolation data must be provided,
- efficiency strongly depends on parameters (ideally **want** dim 2 **and** $N = 2^a$).

7. Isogeny-based cryptography 2.0

Example application: a signature scheme [Ler23].

$$E \xrightarrow{\tau_{\mathrm{sk}}} E_A$$

$P, Q \in E_A[\ell]$ large prime

Sign:

- compute $R = P + \mathrm{H}(E_A, \text{message}) \cdot Q$,
- using knowledge of $\mathrm{End}(E_A)$, provide interpolation data for $\varphi: E_A \to E_A/\langle R \rangle$.

this is the **signature**

Verify:

- compute $R = P + \mathrm{H}(E_A, \text{message}) \cdot Q$,
- check that $\varphi(R) = \infty$ using isogeny interpolation.

7. Isogeny-based cryptography 2.0

Mother(s) of applications: HD variants [BDD+24,DF24,DLR+24,NO24] of **SQIsign** [DKL+20].

very compact signature scheme (on par with ECC),
submitted to renewed NIST competition

$$\begin{tikzcd} E \arrow[r, "\tau_{\mathrm{sk}}"] \arrow[d, "\psi"'] & E_A \arrow[d, "\varphi"] \\ E_{\mathrm{com}} \arrow[r, dashed, "\text{response}"] & E_{\mathrm{ch}} \end{tikzcd}$$

ad-hoc security assumption
slow and hard to scale

Original: respond with smoothening of $\varphi \circ \tau_{\mathrm{sk}} \circ \hat{\psi} : E_{\mathrm{com}} \to E_{\mathrm{ch}}$ through "**generalized** [KLP+14]".

7. Isogeny-based cryptography 2.0

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~~ad-hoc security assumption~~

~~slow and hard to scale~~

Original: respond with smoothening of $\varphi \circ \tau_{\mathrm{sk}} \circ \hat{\psi} : E_{\mathrm{com}} \to E_{\mathrm{ch}}$ through "generalized [KLP+14]".

HD: respond with interpolation data for random bounded-degree isogeny $\sigma\colon E_{\mathrm{com}} \to E_{\mathrm{ch}}$.

7. Isogeny-based cryptography 2.0

The Nakagawa-Onuki trick [NO23] :

$$\begin{tikzcd} F \\ E \arrow[u, "\psi"] \end{tikzcd} \qquad \deg\psi = q?$$

7. Isogeny-based cryptography 2.0

The Nakagawa-Onuki trick [NO23] :

$$\begin{tikzcd} F \\ E \arrow[u, "\psi"] \arrow[u, bend right, "2^a"'] \end{tikzcd}$$

old method: compute I_ψ of norm q, smoothen using [KLP+14]

7. Isogeny-based cryptography 2.0

The Nakagawa-Onuki trick [NO23] :

$$\begin{tikzcd}
F \arrow[r, "\varphi"] & E \\
E \arrow[u, "\psi"] \arrow[ur, "\theta"] \arrow[r, "\varphi'"'] & F' \arrow[u, "\psi'"']
\end{tikzcd}$$

trick: generate $\theta \in \mathrm{End}(E)$ of degree $q(2^a - q)$

7. Isogeny-based cryptography 2.0

The **Nakagawa—Onuki trick** [NO23] :

$$\begin{tikzcd} F \arrow[r, "\varphi"] \arrow[d, "\hat{\psi}"'] & E \arrow[d, "\hat{\psi}'"] \\ E \arrow[r, "\varphi'"'] & F' \end{tikzcd}$$

recover F and interpolation data for ψ

Kani:

$$\Phi : E \times E \xrightarrow{\begin{pmatrix} \psi & \hat{\varphi} \\ -\varphi' & \hat{\psi}' \end{pmatrix}} F \times F'$$

$$\left\{ \left(qQ, \theta(Q)\right) \,\middle|\, Q \in E[2^a] \right\} \longrightarrow \textbf{known!}$$

$$\shortparallel$$

has kernel $\left\{ \left(\hat{\psi}(P), \varphi(P)\right) \,\middle|\, P \in F[2^a] \right\}$

(Generalizes from endomorphism factorization to **isogeny factorization**.)

7. Isogeny-based cryptography 2.0

Clapoti [PR23]: converting an ideal I into an isogeny **without smoothening**.

$$E \xrightarrow[I]{} E_I$$

7. Isogeny-based cryptography 2.0

Clapoti [PR23]: converting an ideal I into an isogeny **without smoothening**.

$$
\begin{tikzcd}
F \arrow[r] & E \arrow[d, "I_2"] \\
E \arrow[u] \arrow[ur, "\theta"] \arrow[r, "I_1"'] & E_I
\end{tikzcd}
$$

$$I_1 \sim I_2 \sim I$$

$$n(I_1) + n(I_2) = 2^a$$

Likewise, compute isogeny

$$\Phi : E \times E \longrightarrow F \times E_I \quad \text{with kernel} \left\{ \left(n(I_2) \cdot Q, \theta(Q)\right) \,\middle|\, Q \in E[2^a] \right\}$$

and extract E_I and interpolation data for φ_{I_1}, then convert into interpolation data for φ_I.

8. Non-cryptographic applications

Other applications [Rob22,KR24]:

- **computing** $\mathrm{End}(E)$ for ordinary $E/\mathbf{F}_q$ in polytime, given factorization of Δ_{Frob_q},

  idea: provide interpolation data for hypothetical $\frac{\mathrm{Frob}_q - \lambda}{m} \in \mathrm{End}(E)$
  run interpolation to check if this is indeed an endomorphism

- **point counting** on $E/\mathbf{F}_{p^n}$ in time $O(n^2 \cdot \mathrm{poly}(\log p))$,

  idea: provide interpolation data for the Verschiebung on E
  study how Kani's endomorphism acts on differentials on $E \times E$
  extract how the Verschiebung acts on on differentials on E

- unconditional $\tilde{O}(\ell^3)$-algorithm for computing **modular polynomial** $\Phi_\ell(X, Y)$.

  see next talk by Sabrina Kunzweiler!

Thanks for sitting this out!

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