Computation of classical and v-adic L-series of t-motives

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Setting

$\mathbb{F}_q$ a finite field with q elements

$A = \mathbb{F}_q[t]$

$K = \mathbb{F}_q(t)$

$\mathfrak{p}, v$ place, *i.e.*

- an irreducible polynomial
- or ∞

$A_\mathfrak{p}$ the completion of A at $\mathfrak{p}$
$K_\mathfrak{p}$ the completion of K at $\mathfrak{p}$
$C_\mathfrak{p}$ the completion of $\bar{K}_\mathfrak{p}$ at $\mathfrak{p}$

Uniformization of "tori"

Let Λ be a discrete A-submodule of C_∞
Then

$$C_\infty/\Lambda \xrightarrow[\sim]{\exp_\Lambda} C_\infty$$

Drinfeld modules

A *Drinfeld module* is a morphism of rings

$$\phi : A \longrightarrow K\{\tau\}$$
$$a \longmapsto \phi_a$$

skew polynomials ($K\{\tau\}$)
$\tau c = c^q \tau$

L-series

a-torsion: $\phi[a] = \{\, x \in \bar{K},\ \phi_a(x) = 0 \,\}$

It is a free module over A/a with an action of $G_K = \mathrm{Gal}(\bar{K}/K)$

Tate module: $T_{\mathfrak{p}}(\phi) = \varprojlim_n \phi[\mathfrak{p}^n]$

It is a free module over $A_{\mathfrak{p}}$ with an action of G_K

local factors: $P_{\mathfrak{p}}(\phi; T) = \det\left(\mathrm{id} - T^{\deg \mathfrak{p}}\, \mathrm{Frob}_{\mathfrak{p}}^{-1} \mid T_\ell(\phi)^{I_{K_{\mathfrak{p}}}}\right)$

$P_{\mathfrak{p}}(\phi; T, s) = \det\left(\mathrm{id} - T^{\deg \mathfrak{p}}\, \mathrm{Frob}_{\mathfrak{p}}^{-1} \mid T_\ell(\phi)(s)^{I_{K_{\mathfrak{p}}}}\right)$ ← Carlitz twist

$I_{K_{\mathfrak{p}}} \subset G_{K_{\mathfrak{p}}} = \mathrm{Gal}(\bar{K}_{\mathfrak{p}}/K_{\mathfrak{p}})$ is the inertia group at

ℓ is a finite place different from $\mathfrak{p}$

L-series: $L(\phi; T) = \prod_{\mathfrak{p} \neq \infty} P_{\mathfrak{p}}(\phi; T)^{-1} \in K[[T]] \subset K_\infty[[T]]$

$$L_v(\phi; T) = \prod_{\mathfrak{p} \neq v, \infty} P_{\mathfrak{p}}(\phi; T)^{-1} \in K[[T]] \subset K_v[[T]]$$

t-motives

From now on, $K = \mathbb{F}_q(\theta)$

Let $\phi : A \to K\{\tau\}$

The motive $\mathrm{M}(\phi)$ of ϕ is $K\{\tau\}$ with

- action of A: $a \bullet m = m\phi_a$
- action of K: $\lambda \bullet m = \lambda m$
- action of τ: $\tau_{\mathrm{M}(\phi)} = \tau m$

It is a finite free module over $A \otimes K \simeq K[t] = \mathbb{F}_q(\theta)[t]$

Definition

A t-motive is a finite free $K[t]$-module M endowed with

$$\tau_M : \quad \tau^\star M\left[\tfrac{1}{t-\theta}\right] \xrightarrow{\sim} M\left[\tfrac{1}{t-\theta}\right]$$

$$\begin{array}{rccc} \tau : & K[t] & \to & K[t] \\ & \sum c_i t^i & \mapsto & \sum c_i^q t^i \end{array}$$

$$\tau^\star M = K[t] \otimes_{\tau, K[t]} M$$

Proposition (étale realization)

$$T_{\mathfrak{p}}(\phi) = \left(\mathrm{M}(\phi)^\vee \otimes_{A\otimes K} (A_{\mathfrak{p}} \otimes \bar{K})\right)^{\tau=1}$$

Definition (Tate module)

$$T_{\mathfrak{p}}(M) = \left(\mathrm{M} \otimes_{A\otimes K} (A_{\mathfrak{p}} \otimes \bar{K})\right)^{\tau=1}$$

L-series of t-motives

Tate module: $$T_{\mathfrak{p}}(M) = \left(\mathrm{M} \otimes_{A\otimes K} (A_{\mathfrak{p}} \otimes \bar{K})\right)^{\tau=1}$$

local factors: $$P_{\mathfrak{p}}(\phi; T) = \det\left(\mathrm{id} - T^{\deg \mathfrak{p}}\,\mathrm{Frob}_{\mathfrak{p}}^{-1} \mid T_{\ell}(\phi)^{I_{K_{\mathfrak{p}}}}\right)$$

$$\begin{aligned} P_{\mathfrak{p}}(M; T, s) &= \det\left(\mathrm{id} - T^{\deg \mathfrak{p}}\,\mathrm{Frob}_{\mathfrak{p}}^{-1} \mid T_{\ell}(M)(s)^{I_{K_{\mathfrak{p}}}}\right) \\ &= \det\left(\mathrm{id} - T^{\deg \mathfrak{p}}\,\mathrm{Frob}_{\mathfrak{p}}^{-1} \mid T_{\ell}(M(s))^{I_{K_{\mathfrak{p}}}}\right) \end{aligned}$$

L-series: $$L(M; T) = \prod_{\mathfrak{p}\neq\infty} P_{\mathfrak{p}}(M; T)^{-1} \qquad \in K[[T]] \subset K_{\infty}[[T]]$$

$$L_v(M; T) = \prod_{\mathfrak{p}\neq v,\infty} P_{\mathfrak{p}}(M; T)^{-1} \qquad \in K[[T]] \subset K_v[[T]]$$

Models of t-motives

Let (M, τ_M) be a t-motive
Recall that it is a finite free module over $A \otimes K = \mathbb{F}_q(\theta)[t]$

Set $R = \mathbb{F}_q[\theta]$
We will work over the tensor product $A \otimes R$; note that it is $\mathbb{F}_q[\theta, t]$

A model of M is a finitely generated $(A \otimes R)$-submodule $N \subset M$ such that

$$\tau_M(\tau^\star N) \subset N\left[\tfrac{1}{t-\theta}\right]$$

Theorem

(i) There exists a unique *maximal* model of M, denoted by $M_{\mathcal{O}}$
It is finite free over $A \otimes R$

(ii) $P_{\mathfrak{p}}(M; T) = \det_{A[\frac{1}{\mathfrak{p}(t)}]}\left(\mathrm{id} - T\tau_M \,\middle|\, M_{\mathcal{O}}\left[\tfrac{1}{\mathfrak{p}(t)}\right]/\mathfrak{p}(\theta)\right)$

Anderson's formula

Define $\Omega^1_R = k[\theta]\cdot d\theta$ and the Cartier operator

$$C : \quad \begin{array}{ccc} \Omega^1_R & \longrightarrow & \Omega^1_R \\ \sum_i a_i \theta^i d\theta & \longmapsto & \sum_i a_{qi+q-1}\theta^i d\theta \end{array}$$

$$\theta^{nq-1} d\theta \longmapsto \theta^{n-1} d\theta$$

Let (M, τ_M) be a t-motive
and $M_{\mathcal{O}}$ be its maximal model

Set $M^\star_{\mathcal{O}} = \mathrm{Hom}_{A\otimes R}\big(M_{\mathcal{O}}, A \otimes \Omega^1_R\big)$

$$\begin{array}{ccc} M_{\mathcal{O}} & \xrightarrow{\tau^\star_M(f)} & A \otimes \Omega^1_R \\ {\scriptstyle \tau_M} \downarrow & & \uparrow {\scriptstyle \mathrm{id} \otimes C} \\ M_{\mathcal{O}} & \xrightarrow{f} & A \otimes \Omega^1_R \end{array}$$

Theorem (Anderson, Böckle)

$$L(M; T) = \det_{K_\infty}\left(\mathrm{id} - T\tau^\star_M \,\middle|\, M^\star_{\mathcal{O}} \otimes_A K_\infty\right)$$

$$L_v(M; T) = \det_{A_v}\left(\mathrm{id} - T\tau^\star_M \,\middle|\, M^\star_{\mathcal{O}}\left[\tfrac{1}{v(\theta)}\right] \otimes_A A_v\right)$$

The spaces are not finite free!
Extreme care is needed
to define the determinants

First algorithm: computation of the maximal model

Input: a t-motive (M, τ_M)

Output: its maximal model $M_{\mathcal{O}}$

The general strategy is similar to *Round 2* algorithm:
we maximize prime by prime, *i.e.* place by place

A key ingredient is the notion of discriminant of a model;
it controls the progression of the algorithm and will eventually ensure termination

$$\Delta_N(\theta) = \mathrm{Fitting}\Big(\mathrm{coker}\ \tau^\star N\big[\tfrac{1}{t-\theta}\big] \xrightarrow{\tau_M} N\big[\tfrac{1}{t-\theta}\big]\Big)$$

A difficulty is that $A \otimes R = \mathbb{F}_q[\theta, t]$ is not a Dedekind domain

Second algorithm: computation of the L-series

Input: a model N of a t-motive
a (finite) place v, a target precision

Output: $\det_{A_v}\left(\mathrm{id} - T\tau_M^\star \,\middle|\, N^\star\left[\frac{1}{v(\theta)}\right] \otimes_A A_v\right)$ it is the L-series if N is maximal

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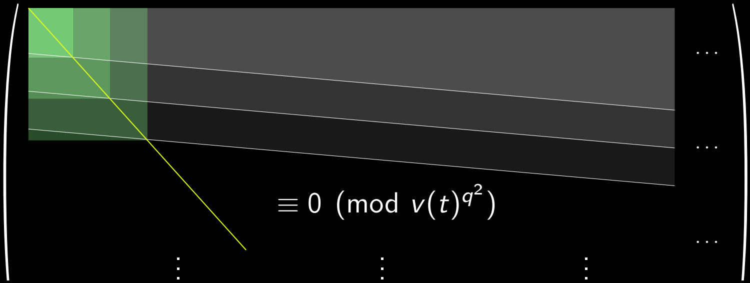
The main ingredient is the construction of an explicit basis of $N^\star\left[\frac{1}{v(\theta)}\right] \otimes_A A_v$ in which the matrix of $\tau_M^\star$ has a very convenient shape



Theorem

The algorithm exhibits quasi-linear complexity in the precision!

Consequences

Theorem

For any place v, the L-series $L_v(M;T) \in A_v[[T]]$ has infinite radius of convergence
In particular, it can be evaluated at $T=1$

Theorem

For any finite place v of degree d, and any positive integer c:

$$s \equiv s' \pmod{q^c(q^d-1)}$$
$$\implies \quad L_v(M;T,s) \equiv L_v(M;T,s') \pmod{v^{q^c}}$$

Conjecture

The order of vanishing at $T=1$ of $L_v(M;T)/P_v(M;T)$
is independant of the finite place v

Thanks

for your attention