An Almost Linear Time Algorithm Testing Whether the Markoff Graph Modulo p is Connected

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github.com/colbyaustinbrown/libbgs

The Markoff Equation

$$x^2 + y^2 + z^2 - xyz = 0$$

- What are the positive integer solutions?
- $(0, 0, 0)$ is the trivial solution
- $(3, 3, 3)$ is also a solution
- All other solutions can be iteratively constructed from $(3, 3, 3)$.

The Markoff Equation: Finding Solutions

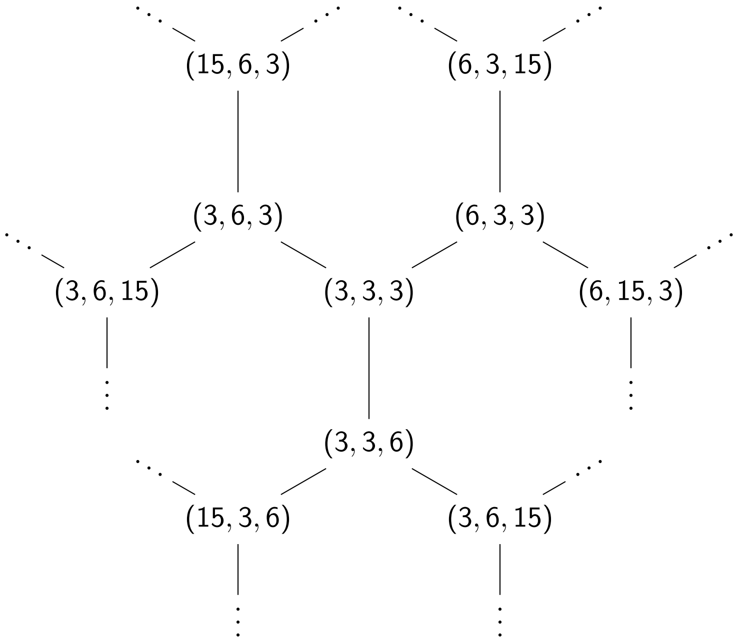
If (a, b, c) is a solution, then so are:

$$V_1(a, b, c) = (bc - a, b, c)$$
$$V_2(a, b, c) = (a, ac - b, c)$$
$$V_3(a, b, c) = (a, b, ab - c)$$

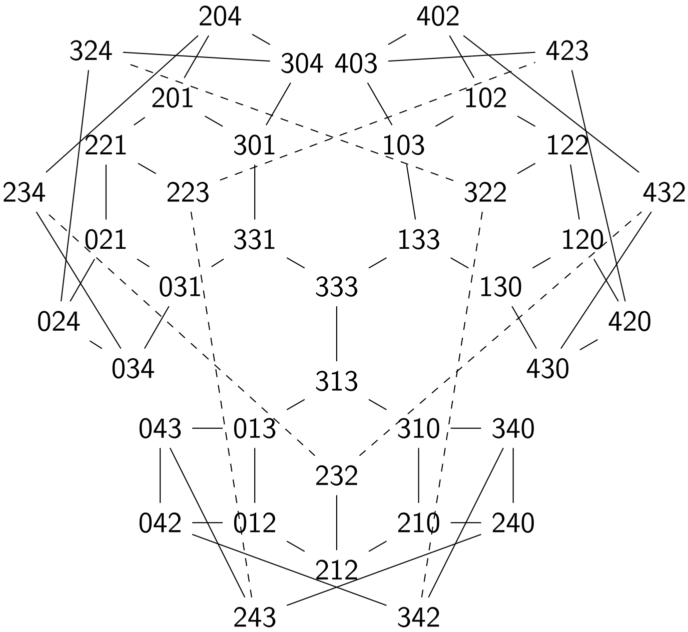
These are the *Vietta involutions*.

- Only fixed point is $(0, 0, 0)$.
- All solutions are obtainable via Vietta involutions.
- Solutions form an infinite 3-regular tree rooted at $(3, 3, 3)$.

The Markoff Tree



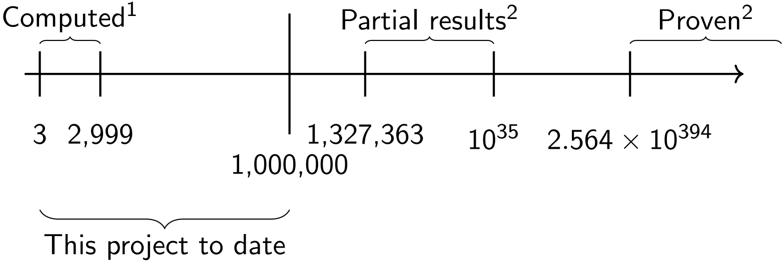
The Markoff Graph modulo 5



Current Progress on Connectivity

Conjecture (Baragar 1990)

The Markoff graph modulo p is connected for every prime p. Equivalently, every solution to the Markoff equation in $\mathbb{F}_p$ lifts to $\mathbb{Z}$.



[1]De Courcy-Ireland and Lee

[2]Eddy, Fuchs, Litman, Martin, Tripeny, and Vanyo

The BGS Argument: Overview

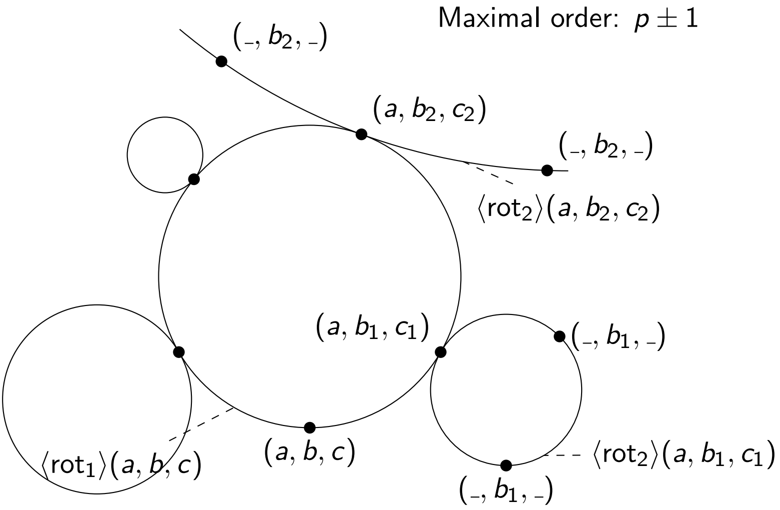
Bourgain, Gamburd, and Sarnak's argument:

1. Consider an action fixing one coordinate:

$$\mathrm{rot}_1(a) : (a, b, c) \mapsto (a, c, bc - a).$$

2. Calculate the rot_1 order of the orbit on a triple (a, b, c).
3. Classify triples by order, which determines connectivity to a "large component."
4. Count the "bad triples" which may not be connected to the large component; all components have size divisible by p [Chen], so there may not be enough.

The BGS Argument: Orbit Intersections



The BGS Argument: Component Sizes

$$\mathrm{rot}_1(a) = (a, c, ac - b)$$

Starting from (a, b, c), orbit of $\mathrm{rot}_1(a)$ is

$$\left(a,\ \alpha\chi^{\ell} + \frac{k}{\alpha\chi^{\ell}},\ \alpha\chi^{\ell+1} + \frac{k}{\alpha\chi^{\ell+1}}\right)$$

where $a = \chi + \chi^{-1}$, and $\mathrm{ord}(\chi) \mid (p \pm 1)$.
Is (a, b, c) connected to the large component? Based on the size of the orbit:

- **Exactly $p \pm 1$:** Yes.
- **At least $p^{\frac{1}{2}+\delta}$:** Yes. Orbit contains a triple of order $p \pm 1$.
- **At least p^{ϵ}:** Yes. Orbit contains a triple of larger order; repeat.
- **Otherwise:** Maybe not. These are the “bad triples”; if there are at least $4p$ of them, the BGS algorithm is inconclusive.

Algorithm Implementing BGS

For each $d|(p \pm 1)$ and $d < L_p$:

1. Calculate all elements $\chi \in \mathbb{F}_{p^2}^{\times}$ with order $\text{ord}_p(\chi + \chi^{-1}) = |\chi| = d$.
2. Let $a = \chi + \chi^{-1}$. Pick the **smaller** of the two strategies:
   - 2.1 Loop through (a, b) pairs and calculate valid c's. If $\text{ord}_p(c) < L$, then the triple (a, b, c) is **bad**.
   - 2.2 For each coset of $\langle \chi \rangle$ pick a representative r and let

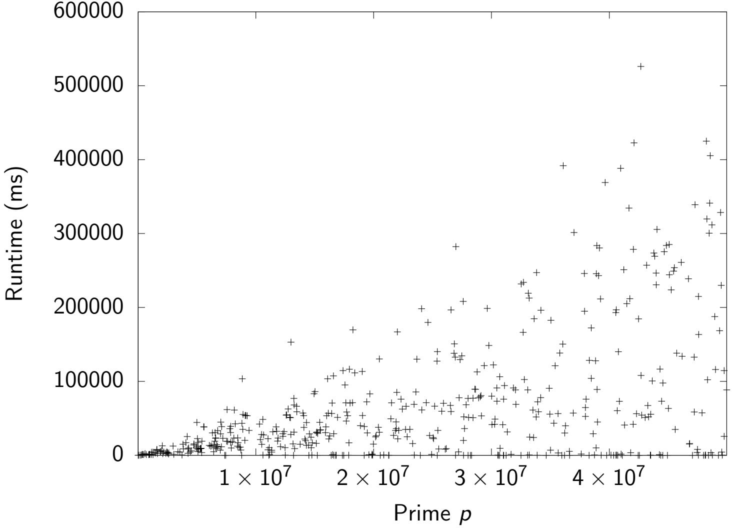
$$b = \frac{\chi + \chi^{-1}}{\chi - \chi^{-1}} \left(r + r^{-1}\right).$$

     Calculate some (fixed) number of valid c values. If $\max(\text{ord}_p(b), \text{ord}_p(c)) < L$ for them all, then **all** triples in the orbit are bad. Otherwise they are all good.
3. Count the bad triples; if there are less than $4p$ of them, the graph is connected. Otherwise, inconclusive.

Results: Connectivity

1. Algorithm implemented in Rust, available at github.com/colbyaustinbrown/libbgs.
2. Graph is connected for all primes $p < 1,000,000$.
3. Tested random sample of primes $p < 50,000,000$; all graphs connected for these, too.
4. Algorithm runs in $o(p^{1+\epsilon})$ time for all $\epsilon > 0$.

Results: Runtime



Measured on 11th Gen Intel Core i5-11320H processor.

Future Work

- Run algorithm on larger range of primes.
- Bound diameter of graphs.
- Conjecture: These graphs form a family of expanders.
- Generalize to other Markoff-like equations.

Thank you!

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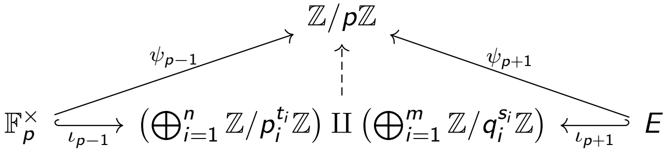
Postscript: Representation of χ

Let $|\chi| | (p-1)$ and $p-1 = p_1^{t_1} \cdots p_n^{t_n}$. Fix a $\mathbb{Z}$-basis of $\mathbb{F}_p^\times$ of the form $\{g_i\}_{i=1}^n$, where $|g_i| = p_i^{t_i}$. Representation of χ for $|\chi| = p_n^{d_n} \cdots p_n^{d_n}$ is:

$$\iota_{p-1} \colon \mathbb{F}_p^\times \to \bigoplus_{i=1}^n \mathbb{Z}/p_i^{t_i}\mathbb{Z}$$

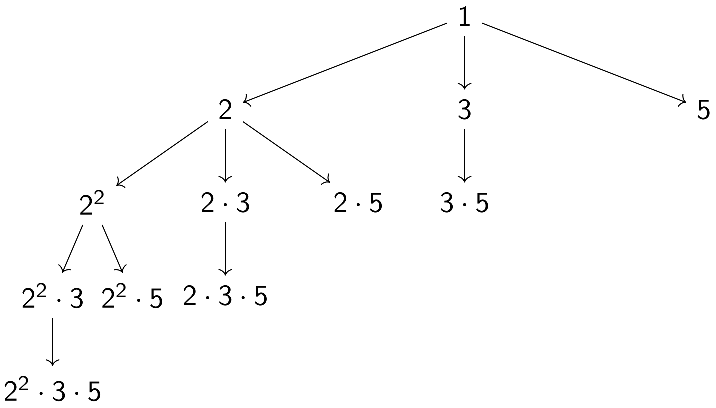
$$\prod_{i=1}^n g_i^{r_i} \mapsto (r_1, \ldots, r_n).$$



where $\psi_{p\pm1}(\chi) = \chi + \chi^{-1}$ and E is the norm-1 elements of $\mathbb{F}_{p^2}^\times$.

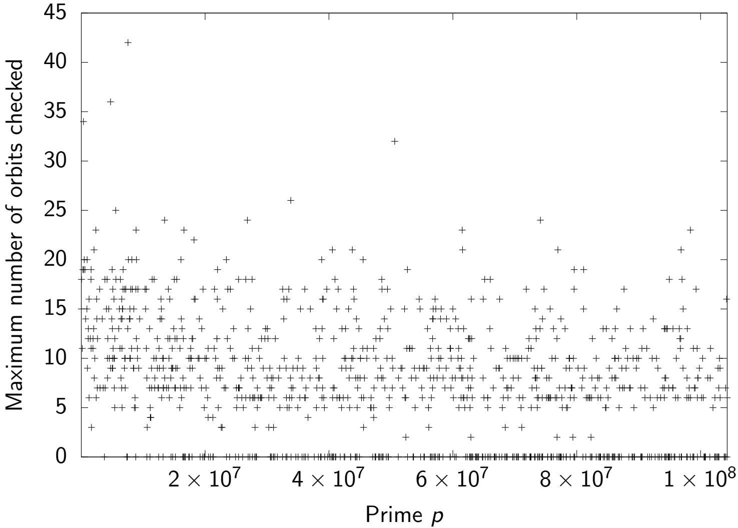
Postscript: Generating χ values

Integer arrays are recursively propogated down a "factor trie":



The factor trie for $n = 60$.

Postscript: Required Checks for Step 2.2



Bibliography

[1] Jean Bourgain, Alexander Gamburd, and Peter Sarnak. "Markoff Surfaces and Strong Approximation: 1". 2016. arXiv: 1607.01530 [math.NT].

[2] William Chen. "Nonabelian level structures, Nielsen equivalence, and Markoff triples". In: *Annals of Mathematics* (forthcoming).

[3] Matthew De Courcy-Ireland and Seugjae Lee. "Experiments with the Markoff Surface". In: *Experimental mathematics* 31.3 (2022), pp. 814–829.

[4] Jillian Eddy et al. "Connectivity of Markoff Mod-P Graphs and Maximal Divisors". 2023. arXiv: 2308.07579 [math.NT].