REDUCTION OF PLANE QUARTICS AND DIXMIER-OHNO INVARIANTS

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Abstract. We characterise, in terms of Dixmier-Ohno invariants, the types of singularities that a plane quartic curve can have. We then use these results to obtain new criteria for determining the stable reduction types of non-hyperelliptic curves of genus 3.

1. Introduction

Let C be a geometrically connected smooth projective curve of genus 3. If its canonical divisor is very ample, so that the curve is non-hyperelliptic, the embedding it defines produces a plane quartic, which is unique up to projective transformations. These simple facts give rise to a remarkable connection between the moduli of curves of genus 3 and the algebra of Dixmier-Ohno invariants for the natural action of SL_3 on the vector space $\mathcal{F}_{3,4}$ of homogeneous polynomials of degree 4 in 3 variables, that is, ternary quartics.

In a landmark paper [DM69], Deligne and Mumford proved that the moduli space $\mathcal{M}_g$ of smooth projective curves of genus $g \geq 2$ is always irreducible, independently of the characteristic of the field of definition. The first of the two proofs they give consists in compactifying this space by adding the curves with mild singularities, the stable curves. Although potentially complicated, blowing up singular loci and taking normalisations makes it theoretically possible to stabilise a projective curve [HM98], from which it is then easier to obtain related quantities of a geometric or arithmetic nature (genus, conductor, *etc*).

At the same time, Mumford was developing geometric invariant theory (GIT), with the aim of explicitly constructing moduli spaces as quotients of parametrising schemes [Mum65]. An important notion in GIT is that of stability. In genus 3 [Art09, Lemma 1.4], a quartic is GIT-stable if it has at most ordinary nodes and cusps. In particular, quartics with non-zero discriminants, *i.e.* smooth, are GIT-stable. On the other hand, a quartic is GIT-unstable, *i.e.* it has a triple point or consists of a cubic and an inflectional tangent, if its Dixmier invariants are all zero. If at least one invariant is non-zero, a quartic is GIT-semi-stable. Quartics with a tacnode are typical examples of semi-stable quartics that are not stable. In a sense, GIT-semi-stability can also be seen as another compactification of the moduli space of projective curves.

A modest contribution to this area is made in this paper for the curves of genus 3. It first aims to classify GIT-semi-stable quartics according to their invariants (see Thm. 3.2). Unlike the case of the action of $\mathrm{SL}(2,\mathbb{C})$ on binary forms of small degree (see for instance [Mes91] for the degree 6 case), little is known from this viewpoint for quartics. We start from the stratification over the complex numbers of singular quartics. The stratification which is based on the different types of singularities a plane quartic can have was studied by Arnol′d in [Arn74; Arn72] and has its origin in work by Du Val on del Pezzo surfaces in [Val34]. This stratification has also been studied by Hui in [Hui79] in great detail, and since we use the formulas from Hui's thesis, we will refer to this stratification as the Hui stratification. This complete classification, that we extend to positive characteristic (greater than 7), consists of 21 possibilities for irreducible quartics, and 34 possibilities for non-reducible ones (see Sec. 2.2, Tab. 2.1 and Tab. 2.2). The subset of GIT-semi-stable quartics, the quartics of interest here, reduces to 33 cases (see the specialisation graph on Fig. 2.1).

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This work allows us, in the second part, to efficiently determine the stable reduction of a plane quartic curve based on its invariants. In genus 2, for example, Mestre exhibits such a connection at the end of [Mes91], and this question is completely resolved in Liu's remarkable work [Liu93]. More generally, the canonical isomorphism between hyperelliptic curves and binary forms is extended as an holomorphic map from the Deligne-Mumford compactification to the compactification of binary forms by adding the singular ones [AL02], and the determination of the stable reduction is determined in terms of the degenerations modulo powers of p of the binary forms ("Cluster picture") in [DDMM19]. This characterisation suggests that the stable reduction type can be, in general, read from hyperelliptic curves invariants valuations (*e.g.* [Lor22, Thm. 6.5]): this is work in progress by Cowland Kellock [Cow24].

In this paper, we endeavour to generalise this approach to the case of plane quartic curves using Dixmier-Ohno invariants. In genus 3, however, the situation is more complicated. There are 42 possibilities for the type of stable reduction, see [Bom+23b]. In *loc. cit.* there is a conjectural correspondence between the stable reduction type of a plane quartic curve and the degeneration type ("Octad picture") of any of its Cayley octads. We also mention here the work of Dokchitser [Dok21] that allows to determine the stable model from a fan associated to the Newton polygon of a plane curve, so long as the curve satisfies a condition known as "Δ_v-regularity". But invariant-based criteria are far more effective in their field of application. Partial results of this form can be found in [LLLR21], characterising good hyperelliptic reduction and in [Bou+21] for the special family of Ciani plane quartics. Our interpretation in terms of p-adic valuations of the invariant-based classification for singularities allows us to give more precise results (see Thm. 5.2 and Cor. 5.5). They are summarised in Tab. 5.1. From the reduction of the invariants we are able to determine all possibilities for the stable reduction type of a curve with such invariants. We demonstrate their usefulness at the end of the article, on a large database of genus 3 curves [Sut19].

Overall, the paper focuses thus on the relationship between GIT and Deligne-Mumford compactifications, for plane quartics via the Hui stratification (see Fig. 1.1). The arrows in the diagram give relationships between the boundary components of the different moduli spaces, *i.e.* if a curve after reduction ends up on one of the boundary components of one of the spaces, then the respective Theorem or Corollary would tell on which of the boundary components of the other moduli space the reduction of the curve could lie. For example, relations satisfied by Dixmier–Ohno invariants of a plane quartic tell us to which stratum of the Hui stratification it belongs. In turn, Hui normal forms for quartics in a given stratum are close enough to the stable model that a significant amount of information about the stable reduction type can be obtained.

$$\begin{tikzcd}
\mathcal{F}_{3,4} \arrow[r] \arrow[d, two heads] & \overline{\mathcal{F}}_{3,4}^{\mathrm{Hui}} \arrow[d, dashed, <->, "\textit{Thm. 3.2}"'] \arrow[dd, dashed, <->, bend left, "\textit{Thm. 5.2}"] \\
\mathcal{F}_{3,4}^{\mathrm{smooth}}/\,\mathrm{SL}_3 \arrow[r] \arrow[d, "\wr"] & \overline{\left(\mathcal{F}_{3,4}^{\mathrm{smooth}}/\,\mathrm{SL}_3\right)}^{\mathrm{GIT}} \arrow[d, dashed, <->, "\textit{Cor. 5.5}"'] \\
\mathcal{M}_3^{\mathrm{nh}} \subseteq \mathcal{M}_3 \arrow[r] & \overline{\mathcal{M}}_3^{\mathrm{DM}}
\end{tikzcd}$$

FIGURE 1.1. Relations among different quartic compactifications

Structure of the paper. Sec. 2 contains the necessary prerequisites about Dixmier-Ohno invariants (Sec. 2.1), Hui's stratification of singular quartics (Sec. 2.2) and stable reduction (Sec. 2.3).

In Sec. 3, we then design an algorithm, Alg. 1, which, given the invariants of a quartic as input, returns which singularities it might have. The algorithm uniquely determines the singularities for quartics all of whose singularities are nodes or cusps, irreducible or not (15 cases). For less common quartics with more complex singularities, the invariants no longer allow a precise determination. For example, Alg. 1 cannot tell whether a quartic has just a single tacnode, or a tacnode together with an additional node or cusp.

Its validity is based on a sequence of technical propositions given in Sec. 3.2. Our approach follows a "guess and prove" paradigm. Relations stated in these propositions have been calculated heuristically over $\mathbb{Q}$, by interpolating them on normalised quartics given by Hui (see tables in Sec. 2.2). We independently prove that they are valid in Sec. 4. These make use of the computational algebra system MAGMA [BCP97] and the interested reader can download the corresponding script at [Bom+23a, file `G3SingularProof.m`]. Using MAGMA version 2.28-8, these computations need about 6 GB of memory and less than two hours on a standard laptop (INTEL I7-8850H CPU).

In Sec. 5 we relate the GIT compactification of $\mathcal{M}_3$ with the Deligne-Mumford compactification of $\mathcal{M}_3$ by stable curves. This is done via the Hui stratification. In Sec. 5.1 we establish Thm. 5.2 stating a relation between stable reduction types and singularities of GIT-semi-stable plane quartics. Its proof is postponed to Sec. 5.2. We translate these results into explicit reduction criteria in Corollary 5.5. These characterisations are tight, *i.e.* one-to-one, for quartics whose reduction has only nodal singularities. For more complex singularities, the stable reduction type is not unique. We determine the different possibilities and we prove that all of them may be achieved. We conclude in Sec. 5.3 with an application to integer quartics of small discriminant.

Remark 1.1. In Sections 3 and 4, we state theorems, propositions, and lemmas only in fields of characteristic 0. A primary rationale for this limitation is that Dixmier-Ohno invariants do not constitute a complete list of generators for the equivalence of ternary quartic forms under the action of SL_3 in fields of positive characteristic $p = 2$, 3, 5 and 7. It is also uncertain at the time of writing whether they generate the invariant algebra for $p > 7$ (see Rem. 2.1). A second reason for this limitation is that their proofs often rely on Gröbner basis computations performed over $\mathbb{Q}$ (*e.g.* Prop. 3.5 to Prop. 3.8, see Rem. 3.11).

However, we expect that these statements hold in positive characteristic, excluding $p = 2$, 3, 5 and 7. In practice, one can verify a particular statement for a particular value of p by running the same computational verification in characteristic p. This leads us to the conjectural result Conj. 3.3.

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2. Preliminaries

This section recalls the basic definitions and results on invariant theory, singular quartics and stable reduction needed for this work.

2.1. **Invariants.** In [Dix87], Dixmier gave a list of 7 homogeneous polynomial invariants for the equivalence of ternary quartic forms under the action of $\mathrm{SL}_3(\mathbb{C})$, denoted I_3, I_6, I_9, I_{12}, I_{15}, I_{18} and I_{27}. The last invariant satisfies the equality $2^{40} I_{27} = D_{27}$ where D_{27} is the discriminant of the form. This list was later completed by Ohno with J_9, J_{12}, J_{15}, J_{18}, I_{21} and J_{21} ([Ohn07, Theorem 4.1], see also [Els15]) into a list of 13 homogeneous generators of the $\mathbb{C}$-algebra of invariants. These invariants are defined over $\mathbb{Z}[\frac{1}{2\cdot 3}]$, and are now called the *Dixmier-Ohno invariants*. When at least one of their values at a ternary quartic form F is not zero, we denote by $\mathrm{DO}(F)$ the corresponding point in the weighted projective space with weights 3, 6, 9, 9, 12, 12, 15, 15, 18, 18, 21, 21, 27.

This algebra of invariants is finitely generated as a $\mathbb{C}[I_3, I_6, I_9, I_{12}, I_{15}, I_{18}, I_{27}]$-module, or equivalently Dixmier invariants form a so-called "homogeneous system of parameters" (HSOP).

Remark 2.1. In fields of characteristic $p > 0$, HSOPs are also known for the corresponding invariant algebra, even for $p = 3$ if we assume that the huge invariants of degree 54 and 81 given in [LLLR21, Sec. 4] are well defined. In particular, Dixmier invariants give an HSOP for $p > 7$, except for $p = 19$, 47, 277 and 523 where I_9 must be replaced by $I_9 - J_9$. Note that it is not yet proven that adding Ohno invariants to these HSOPs is sufficient to generate the invariants algebras completely when $p > 0$, but it is a commonly held belief that this is the case for $p > 7$. An analogous result for hyperelliptic curves holds [LR12, Prop. 1.9].

2.2. **Singular quartics.** In [Hui79], Hui describes a complete stratification of the space of singular plane quartic curves defined over $\mathbb{C}$. He also provides normal forms up to the action of $\mathrm{SL}_3(\mathbb{C})$ for all singularity types. We summarise his results in Tab. 2.1 and Tab. 2.2. The first column[1] gives the types of the singularities of the quartics given in the third column, according to Arnold's classification [Arn72], *i.e.* $\mathrm{A}_1, \ldots, \mathrm{A}_6$, D_4, D_5, E_6 and E_7. Multiples and powers of these types denote curves with several singularities. For instance, a quartic of type $\mathrm{A}_1^2\mathrm{A}_2$ has three distinct singular points, two of type A_1 and one of type A_2. We prefix by "^{r}" types of curves which are not irreducible. For example, a curve of type ${}^r\mathrm{A}_1^4{}_{cub}$ has two components, a line and a cubic. For quartics with non-isolated singularities (Tab. 2.2b), we use the more explicit notation ℓ and c, for a line and a conic. Thus a quartic of type c^2 is the square of a conic. Otherwise, the second column in these tables gives the number of parameters α, β, … needed to define the normal forms.

Remark 2.2. Often the singular points of Hui normal forms are exactly of the expected type for any choice of parameters α, β *et cetera*. This is the case, for example, for the normal forms of A_1 and A_2 at $(1:0:0)$, or A_1^2 at this point and at $(0:1:0)$. The only way for these quartics to specialise to other singularity types is thus to have additional singular points (we use this fact in Sec. 4.3). Where this is not the case, Hui explicitly states the values of the parameters that change the singularity type. For example, for A_1^3 he indicates that by setting α, β and γ to ± 2, the type of the three singular points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$ becomes A_2.

Remark 2.3. In fact, Hui's manuscript has to be checked with care, but the classification and the normal forms do hold in characteristic $p > 7$ for all p. For smaller p, one of the issues is that the singularities can have another type than those known in characteristic zero. For example, the singular point $(0:0:1)$ of the Hui normal form for A_6, $x^2z^2 + 2\,y^2xz + y^4 - yx^3$, has Milnor number 9 instead of 6 in characteristic 7.

[1] Drawings are for illustrative purposes only, and do not correspond to graphs of Hui normal forms.

Type	#	Normal forms
Smooth [Shi93]	6	$xz^3 + z(\alpha\, x^3 + \beta\, x^2y + y^3) + \gamma\, x^4 + \delta\, x^3y + \epsilon\, x^2y^2 + \zeta\, xy^3 + y^4$
A_1	5	$yz^3 + (\alpha\, y^2 + x^2)z^2 + (\beta\, y^3 + \gamma\, y^2x + yx^2)z + \delta\, y^4 + \epsilon\, y^3x$
A_2	4	$yz^3 + (\alpha\, y^2 + \beta\, yx + x^2)z^2 + (\gamma\, y^3 + \delta\, y^2x)z + y^3x$
A_3	3	$x^2z^2 + \alpha\, y^2xz + y^4 + \beta\, y^3x + \gamma\, y^2x^2 + yx^3$
A_4	2	$x^2z^2 + 2\, y^2xz + y^4 + \alpha\, y^3x + \beta\, y^2x^2 + yx^3$
D_4	2	$(y^2x + yx^2)z + \alpha\, y^4 + \beta\, y^2x^2 - x^4$
A_5	1	$x^2z^2 + 2\, y^2xz + y^4 - y^2x^2 + \alpha\, yx^3$
D_5	1	$yx^2z - y^4 + \alpha\, y^3x - x^4$
A_6	0	$x^2z^2 + 2\, y^2xz + y^4 - yx^3$
E_6	0	$x^3z - y^4 - \alpha\, y^2x^2 \quad (\alpha = 0 \text{ or } 1)$
A_1^2	4	$(y^2 + x^2)z^2 + (\alpha\, yx^2 + \beta\, x^3)z + y^2x^2 + \gamma\, yx^3 + \delta\, x^4$
$\mathrm{A}_1\,\mathrm{A}_2$	3	$y^2z^2 + (\alpha\, yx^2 + x^3)z + y^2x^2 + \beta\, yx^3 + \gamma\, x^4$
A_2^2	2	$y^2z^2 + (\alpha\, yx^2 + x^3)z + yx^3 + \beta\, x^4$
$\mathrm{A}_1\,\mathrm{A}_3$	2	$x^2z^2 + (\alpha\, y^2x + yx^2)z + y^4 + \beta\, y^3x$
$\mathrm{A}_2\,\mathrm{A}_3$	1	$x^2z^2 + \alpha\, y^2xz + y^4 + y^3x$
$\mathrm{A}_1\,\mathrm{A}_4$	1	$x^2z^2 + 2\, y^2xz + y^4 - y^3x + \alpha\, y^2x^2$
$\mathrm{A}_2\,\mathrm{A}_4$	0	$x^2z^2 + 2\, y^2xz + y^4 - y^3x$
A_1^3	3	$(y^2 + \alpha\, yx + x^2)z^2 + (\beta\, y^2x + \gamma\, yx^2)z + y^2x^2$
$\mathrm{A}_1^2\,\mathrm{A}_2$	2	$(y^2 + \alpha\, yx + x^2)z^2 + (\beta\, y^2x - 2\, yx^2)z + y^2x^2$
$\mathrm{A}_1\,\mathrm{A}_2^2$	1	$(y^2 + \alpha\, yx + x^2)z^2 + (-2\, y^2x - 2\, yx^2)z + y^2x^2$
A_2^3	0	$(y^2 - 2\, yx + x^2)z^2 + (-2\, y^2x - 2\, yx^2)z + y^2x^2$

Table 2.1. Irreducible normal forms of singular plane quartic curves

Furthermore, Hui studies precisely how a given stratum specialises in strata of smaller dimensions [Hui79, chap. 7]. We summarise his results on Fig. 2.1 for the strata we are interested in, *i.e.* those which are non-unstable (in the GIT sense), starting from quartics without singularities (top) up to the most singular quartics (bottom). Furthermore, we have grouped strata that are not GIT-stable, *i.e.* with a A_3 singularity or more, into three sets, $V(\mathbf{A_4})$, $V(\mathbf{A_3})$ and $V({}^{\mathbf{r}}\mathbf{A_1}\,\mathbf{A_3})$, which are delimited by dashed lines. In terms of invariants, strata within the same set are indistinguishable from the stratum of smallest dimension.

2.3. **Stable reduction of curves.** Consider a smooth projective and geometrically connected curve C over a local field K. By Deligne and Mumford [DM69], there exists a finite extension

Type	#	Normal forms
${}^{r}\mathrm{A}_5$	1	$x\,(xz^2 + (\alpha\, yx + x^2)z + y^3)$
${}^{r}\mathrm{X}_9(\tilde{\mathrm{A}}_7)$	1	$y^4 + \alpha\, y^2x^2 + x^4$
${}^{r}\mathrm{D}_6$	0	$x\,(z^3 + yxz + x^3)$
${}^{r}\mathrm{A}_7$	0	$(yz + y^2 + x^2)\,(yz + x^2)$
${}^{r}\mathrm{E}_7$	0	$x(z^3 + xz^2 + yx^2)$
${}^{r}\mathrm{A}_1\,\mathrm{A}_3$	2	$x\,(\alpha\, xz^2 + (y^2 + \beta\, yx + x^2)z + y^2x)$
${}^{r}\mathrm{A}_1\,\mathrm{D}_4$	1	$x\,(yz^2 + (yx + \alpha\, x^2)z + x^3)$
${}^{r}\mathrm{A}_3^2$	1	$(yz + x^2)\,(\alpha\, yz + x^2)$
${}^{r}\mathrm{A}_1\,\mathrm{A}_5$	0	$(z^2 + yx)\,(z^2 + yz + yx)$
${}^{r}\mathrm{A}_1\,\mathrm{A}_5$	0	$x\,(z^3 + yxz + y^2x)$
${}^{r}\mathrm{A}_1\,\mathrm{D}_5$	0	$x\,(yz^2 + \alpha\, x^2z + x^3)\quad (\alpha = 0 \text{ or } 1)$
${}^{r}\mathrm{A}_1\,\mathrm{D}_6$	0	$yz\,(xz + y^2)$
${}^{r}\mathrm{A}_2\,\mathrm{A}_5$	0	$x\,(z^3 + y^2x)$
${}^{r}\mathrm{A}_1^3$	3	$x\,((y + \alpha\, x)z^2 + (y^2 + \beta\, yx + \gamma\, x^2)z + y^2x)$
${}^{r}\mathrm{A}_1^2\,\mathrm{A}_3$	1	$x\,(xz^2 + (y^2 + \alpha\, yx)z + y^2x)$
${}^{r}\mathrm{A}_1^2\,\mathrm{A}_3$	1	$((y + x)z + yx)\,((\alpha\, y + x)z + yx)$
${}^{r}\mathrm{A}_1^2\,\mathrm{D}_4$	0	$yz\,((y + x)z + yx)$
${}^{r}\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3$	0	$x\,(xz^2 + (y^2 + 2\, yx)z + y^2x)$
${}^{r}\mathrm{A}_1\mathrm{A}_3^2$	0	$\mathrm{y}z\,(yz + x^2)$
${}^{r}\mathrm{A}_1^4$	2	$x\,((y + \alpha\, x)z^2 + (y^2 + \beta\, yx)z + y^2x)$
${}^{r}\mathrm{A}_1^4$	2	$((y + x)z + yx)\,((\alpha\, y + \beta\, x)z + yx)$
${}^{r}\mathrm{A}_1^3\,\mathrm{A}_2$	1	$x\,(z^3 + \alpha\, yz^2 + y^2z + y^2x)$
${}^{r}\mathrm{A}_1^3\,\mathrm{A}_3$	0	$\mathrm{y}z\,((y + x)z + x^2)$
${}^{r}\mathrm{A}_1^3\,\mathrm{D}_4$	0	$xyz\,(z + x)$
${}^{r}\mathrm{A}_1^5$	1	$\mathrm{y}z((\alpha\, y + x)z + yx + x^2)$
${}^{r}\mathrm{A}_1^6$	0	$xyz\,(z + y + x)$

(a) Non-irreducible

Type	Normal forms
$\ell^2\, c$	$x^2\,((y + x)z + yx)$
$\ell^2\, c$	$z^2\,(xz + y^2)$
$\ell\,\ell\,\ell^2$	xyz^2
$\ell\,\ell\,\ell^2$	$xz^2(z + x)$
c^2	$(z^2 + yx)^2$
$\ell^2\,\ell^2$	x^2z^2
$\ell\,\ell^3$	xz^3
ℓ^4	z^4

(b) Non-isolated

TABLE 2.2. Non-irreducible normal forms of singular plane quartic curves

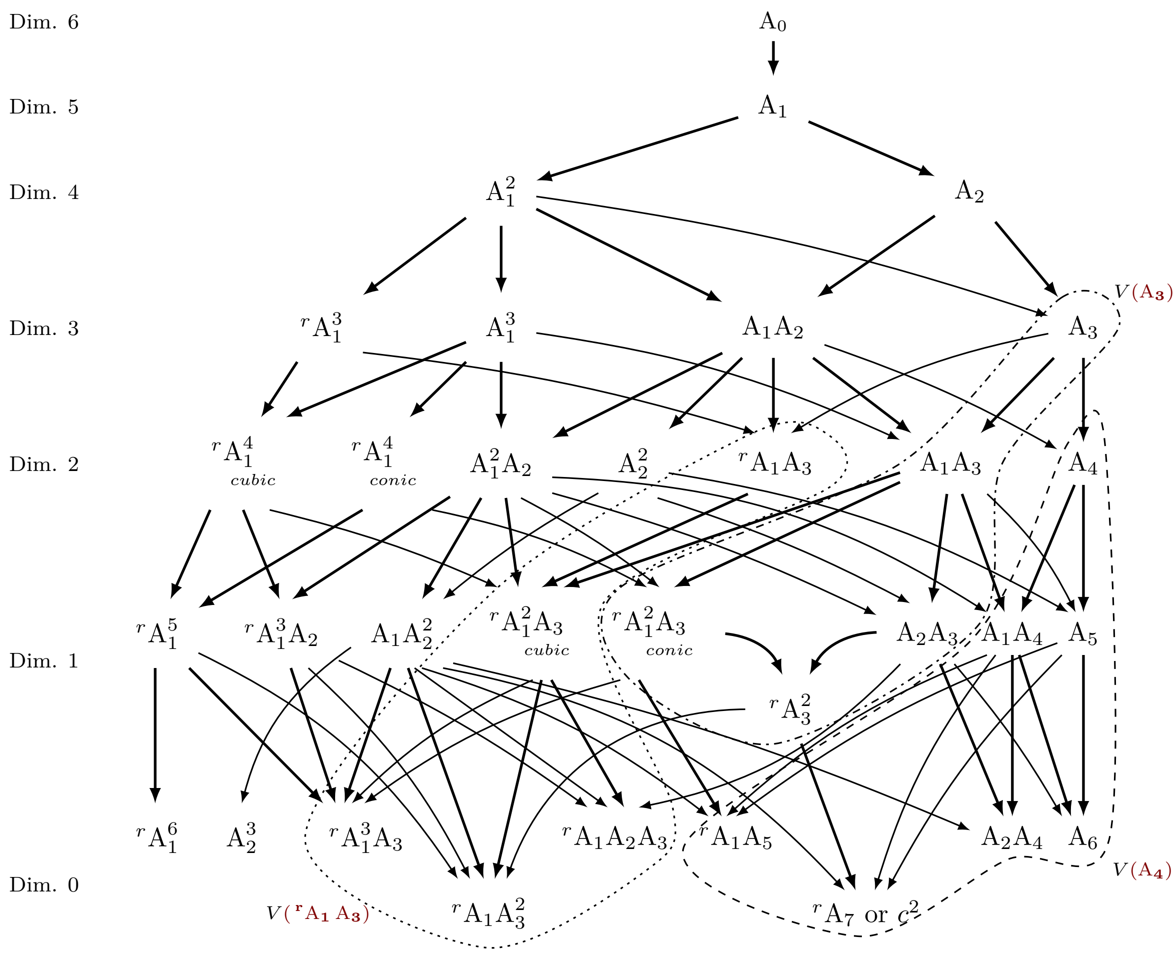

FIGURE 2.1. Singularity specialisations (non-unstable strata)

K' of K such that $C \times_K K'$ is the generic fibre of a stable curve $\mathcal{C}$ over the integral closure of the ring of integers of K inside K'.

The singularities of a stable curve are ordinary double points, and its irreducible components of geometric genus 0 have at least three such double points, counted with multiplicity. The *stable reduction* of a curve is the special fibre of such a stable model, and the *(stable) reduction type* is the stable type of the stable reduction, including whether the reduction is hyperelliptic or not.

It is possible to obtain all stable types recursively by degeneracy. In genus 3, without distinction between hyperelliptic and non-hyperelliptic, this produces a list of 42 types, which we show in Fig. 5.1 of Sec. 5.1. This graph is derived from [Bom+23b]. It lists the quartic stable reduction types ordered by degeneration and codimension from top to bottom according to the stratification of $\overline{\mathcal{M}_3}^{\mathrm{DM}}$ they induce. In addition to small descriptive drawings that represent irreducible components and their intersections with their genus indicated by the thickness of the component, we use a compact naming convention where `0`, `1`, `2` or `3` refer to curves of that genus, `n` means node, `e` means elliptic curve, `m` means multiplicative reduction of elliptic curves, `X=Y` means that `X` and `Y` intersect at two points, `Z` is shorthand for two distinct genus 0 components intersecting at two distinct points or `0=0`, and `X---Y` (resp. `X----Y`) means that `X` and `Y` intersect at 3 (resp. 4) points. The remaining two types are `(CAVE)`, 3 lines intersecting at 5 points, and `(BRAID)`, 4 lines intersecting at 6 points. When the stable reduction is hyperelliptic, we put a `H` as subscript, *e.g.* $(1 = 1)_{\mathtt{H}}$.

3. Singularities and invariant strata of plane quartics

Let F be a (possibly singular) plane quartic defined over an algebraically closed field $\overline{K}$ of characteristic 0. Let $\mathrm{DO}(F) = (I_3 : I_6 : \cdots : I_{27})$ be its Dixmier-Ohno invariants. Our aim is to retrieve from them a list of candidates for the singularity type of F. This amounts to determining in each case the equations satisfied by these invariants. The simplest example of such results is that a quartic is singular if and only if its discriminant, *i.e.* I_{27}, is zero. In the following sections, we will apply some of these results to get information on the reduction of a quartic defined over a non-archimedean local field.

Before going into more detail, we can already notice that it is not possible to derive a tight classification of the singularity types from the invariants. Already, quartics with singularity multiplicities greater than two are unstable, so their invariants are identically zero. For those types that correspond to the grey lines in Tab. 2.1 and Tab. 2.2, the question is thus trivial. In fact, all non-unstable types with an A_4 singularity or more cannot be separated, as well as those of type A_3 and its compounds ($\mathrm{A}_1\mathrm{A}_3$, $\mathrm{A}_2\mathrm{A}_3$, *etc.*), or of type ${}^r\mathrm{A}_1\mathrm{A}_3$ and its compounds (${}^r\mathrm{A}_1^2\mathrm{A}_3$, ${}^r\mathrm{A}_1\mathrm{A}_2\mathrm{A}_3$, *etc.*)

Example 3.1. The plane quartic $x^2z^2 + y^4 + yx^3 = 0$ with a singularity of type A_3 at $(0 : 0 : 1)$ and the plane quartic $x^2z^2 + y^4 + y^3z + y^2z^2 = 0$ with singularities of type A_3 at $(1 : 0 : 0)$ and A_1 at $(0 : 0 : 1)$ have the same Dixmier-Ohno invariants $(18 : 0 : 1944 : 648 : 0 : 11664 : 0 : 0 : 0 : 0 : 0 : 0 : 0)$.

3.1. **The algorithm.** Suppose $\mathrm{char}(K) = 0$. Considering the Dixmier-Ohno invariants as variables, we can define a weighted projective space $\mathbb{P}^{12}_{3,6,\ldots,27}$ of dimension 13. We consider the ideals (A) generated by the relations that they satisfy on the singularity strata A. These are defined in Sec. 3.2. By $V(\mathrm{A})$ we denote the algebraic set defined by these equations. It corresponds to quartics with singularity type A, but also possibly to some other more singular strata below it in the specialisation graph of Fig. 2.1.

The result is an algorithm for characterising singularities, which essentially consists of checking whether the Dixmier-Ohno invariants of a given plane quartic lie in $V(\mathrm{A})$ for the different strata A, starting with the smallest strata. It yields Alg. 1. The following theorem states that this algorithm is correct.

Theorem 3.2. *Let $C\colon F = 0$ be a plane quartic curve over an algebraically closed field $\overline{K}$ of characteristic 0. Then its singularity type is amongst the types returned by Alg. 1 on input* $\mathrm{DO}(F)$.

The proof is based on Prop. 3.4 to Prop. 3.9, given in Sec. 3.2. We conjecture that Thm. 3.2 is also true in positive characteristic p for p not too small (*cf.* Rem. 1.1).

Conjecture 3.3 (positive characteristic hypothesis)**.** *Alg. 1 is valid for any algebraically closed field $\overline{K}$ of characteristic p for all $p \neq 2, 3, 5, 7$.*

In support of this conjecture, we have automated the verification of Prop. 3.4 to Prop. 3.9 with MAGMA and applied it for all primes up to 100. The criterion $p > 7$ is necessary for the list of quartic singularities to be complete (*cf.* Rem. 2.3) and for $\mathrm{DO}(F)$ to be defined differently for $p = 2$, 3, 5 and 7 (*cf.* Rem. 2.1).

Algorithm 1: Plane quartic singularity types

Input : Dixmier-Ohno invariants $(I_3 : I_6 : \cdots : I_{27}) \in \mathbb{P}^{12}_{3,6,\ldots,27}(\overline{K})$ of a plane quartic, char$(\overline{K}) = 0$
Output: A set of possible singularity types

// Easy cases
If $I_{27} \neq 0$, then **return** $\{$ *Smooth* $\}$;
If $(I_3 : I_6 : \cdots : I_{27}) = 0$, then **return** $\{$ *Unstable* $\}$;

// Dimension 0
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{{}^r A_1^6})$, then **return** $\{ {}^r\mathrm{A}_1^6 \}$; // Prop. 3.4
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{A_2^3})$, then **return** $\{ \mathrm{A}_2^3 \}$;
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{A_4})$, then **return** $\{ \mathrm{A}_4,\ \mathrm{A}_5,\ \mathrm{A}_6,\ \mathrm{A}_1\,\mathrm{A}_4,\ \mathrm{A}_2\,\mathrm{A}_4,\ {}^r\mathrm{A}_7,\ {}^r\mathrm{A}_1\,\mathrm{A}_5,\ c^2 \}$;
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{{}^r A_1\, A_3})$, then **return** $\{ {}^r\mathrm{A}_1\,\mathrm{A}_3,\ {}^r\mathrm{A}_1^2\,\mathrm{A}_3,\ {}^r\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3,\ {}^r\mathrm{A}_1\,\mathrm{A}_3^2,\ {}^r\mathrm{A}_1^3\,\mathrm{A}_3 \}$;

// Dimension 1
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{{}^r A_1^5})$, then **return** $\{ {}^r\mathrm{A}_1^5 \}$; // Prop. 3.5
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{{}^r A_1^3\, A_2})$, then **return** $\{ {}^r\mathrm{A}_1^3\,\mathrm{A}_2 \}$;
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{A_1\, A_2^2})$, then **return** $\{ \mathrm{A}_1\,\mathrm{A}_2^2 \}$;
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{A_3})$, then **return** $\{ \mathrm{A}_3,\ \mathrm{A}_1\,\mathrm{A}_3,\ \mathrm{A}_2\,\mathrm{A}_3,\ {}^r\mathrm{A}_3^2,\ {}^r\mathrm{A}_1^2\,\mathrm{A}_3 \}$;

// Dimension 2
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{A_1^2\, A_2})$, then **return** $\{ \mathrm{A}_1^2\,\mathrm{A}_2 \}$; // Prop. 3.6
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{A_2^2})$, then **return** $\{ \mathrm{A}_2^2 \}$;
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{{}^r A_1^4}_{\mathbf{cub}})$, then **return** $\{ {}^r\mathrm{A}_1^4$ *(line and cubic)* $\}$;
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{{}^r A_1^4}_{\mathbf{con}})$, then **return** $\{ {}^r\mathrm{A}_1^4$ *(two conics)* $\}$;

// Dimension 3
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{A_1^3})$, then **return** $\{ \mathrm{A}_1^3 \}$; // Prop. 3.7
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{A_1 A_2})$, then **return** $\{ \mathrm{A}_1\,\mathrm{A}_2 \}$;
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{{}^r A_1^3})$, then **return** $\{ {}^r\mathrm{A}_1^3 \}$;

// Dimension 4
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{A_2})$, then **return** $\{ \mathrm{A}_2 \}$; // Prop. 3.8
If $(I_3 : I_6 : \cdots : I_{27})$ is in $V(\mathbf{A_1^2})$, then **return** $\{ \mathrm{A}_1^2 \}$;

// Dimension 5
return $\{\mathrm{A}_1\}$;

3.2. **Algebraic characterisations.** We present the propositions necessary for the proof of Thm. 3.2 by increasing dimensions. Here, it is still assumed that the field of definition is an algebraically closed field $\overline{K}$ of characteristic zero.

We have to understand "dimension" as the number of variables that parameterise the Hui normal forms for a singularity type stratum (*cf.* Tab. 2.1 and Tab. 2.2). In the following, the singularity loci will be viewed as the zero set of an ideal (A) in the projective space $\mathbb{P}^{12}_{3,6,\ldots,27}$. The dimension of the singularity loci will then become the Krull dimension of the quotient of the coordinate ring of the weighted $\mathbb{P}^{12}$ by the ideal (A), minus one. Since for singularity type A_3 (resp. A_4 and ${}^r\mathrm{A}_1\mathrm{A}_3$), the ideal is of dimension 2 (resp. 1), this singularity is not characterised as one would expect in Prop. 3.7 (resp. Prop. 3.6) with the other singularity types of dimension 3 (resp. 2) , but in Prop. 3.5 (resp. Prop. 3.4) with the singularity types of dimension 1 (resp. 0). In terms of invariants (see Fig. 2.1), it's not possible to distinguish the stratum A_3 (resp. A_4 and ${}^r\mathrm{A}_1\mathrm{A}_3$) from the stratum ${}^r\mathrm{A}_3^2$ (resp. c^2 and ${}^r\mathrm{A}_1\mathrm{A}_3^2$).

For lack of space, and also because of the technical nature of such results, we present these 6 propositions in as compact a form as possible. In particular we do not give the polynomials that define these ideals explicitly here, but they are available at [Bom+23a, file `reductioncriteria.m`]. The proof of these statements is of a computational nature. We summarise its main steps in Sec. 4.

Strata of dimension 0.

Proposition 3.4. *Let C be a plane quartic given by a GIT-semi-stable ternary quartic F over an algebraically closed field $\overline{K}$ of characteristic 0, then C is of singularity type*

- ${}^{r}\mathrm{A}_1^6$ *if and only if*

$$\mathrm{DO}(F) = \left(1 : \frac{-1}{144} : \frac{1}{9} : \frac{-1}{3} : \frac{-4}{81} : \frac{-1}{18} : \frac{-1}{972} : \frac{1}{36} : \frac{1}{243} : \frac{1}{27} : \frac{1}{162} : \frac{-7}{144} : 0\right), \tag{${}^{r}\mathrm{A}_1^6$}$$

- A_2^3 *if and only if*

$$\mathrm{DO}(F) = \left(1 : \frac{-1}{108} : \frac{97}{324} : \frac{-121}{324} : \frac{-325}{2916} : \frac{-47}{324} : \frac{121}{11664} : \frac{7}{1296} : \frac{1595}{104976} : \frac{985}{34992} : \frac{637}{78732} : \frac{-1057}{314928} : 0\right), \tag{A_2^3}$$

- A_4, A_5, A_6, $\mathrm{A}_1\,\mathrm{A}_4$, $\mathrm{A}_2\,\mathrm{A}_4$, ${}^{r}\mathrm{A}_7$, ${}^{r}\mathrm{A}_1\,\mathrm{A}_5$ *or* c^2 *if and only if*

$$\mathrm{DO}(F) = \left(1 : \frac{1}{180} : \frac{49}{36} : \frac{49}{60} : \frac{343}{1620} : \frac{49}{36} : \frac{1715}{3888} : \frac{343}{3600} : \frac{2401}{3888} : \frac{2401}{10800} : \frac{343}{1620} : \frac{2401}{720} : 0\right), \tag{A_4}$$

- ${}^{r}\mathrm{A}_1\,\mathrm{A}_3$, ${}^{r}\mathrm{A}_1^2\,\mathrm{A}_3$, ${}^{r}\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3$, ${}^{r}\mathrm{A}_1\,\mathrm{A}_3^2$ *or* ${}^{r}\mathrm{A}_1^3\,\mathrm{A}_3$ *if and only if*

$$\mathrm{DO}(F) = \left(1 : \frac{-1}{144} : \frac{7}{36} : \frac{1}{6} : \frac{-17}{1296} : \frac{7}{288} : \frac{625}{31104} : \frac{-25}{1152} : \frac{775}{62208} : \frac{-35}{2304} : \frac{-1}{20736} : \frac{1}{256} : 0\right). \tag{${}^{r}\mathrm{A}_1\,\mathrm{A}_3$}$$

Strata of dimension 1.

Proposition 3.5. *Let C be a plane quartic given by a GIT-semi-stable ternary quartic F over an algebraically closed field $\overline{K}$ of characteristic 0 that is not in a stratum of dimension 0, then C is of singularity type*

- ${}^{r}\mathrm{A}_1^5$ *if and only if* $\mathrm{DO}(F)$ *is in the algebraic set* $V({}^{r}\mathrm{A}_1^5)$ *of an ideal* $({}^{r}\mathrm{A}_1^5)$ *defined by 13 polynomials whose degree profile is* 6, 9, 12, 15^2, 18^2, 21^3, 27^2 *and* 36, $\quad({}^{r}\mathrm{A}_1^5)$
- ${}^{r}\mathrm{A}_1^3\,\mathrm{A}_2$ *if and only if* $\mathrm{DO}(F)$ *is in the algebraic set of an ideal* $({}^{r}\mathrm{A}_1^3\,\mathrm{A}_2)$ *defined by 11 polynomials whose degree profile is* 6, 9, 12^2, 15^2, 18^2, 21^2 *and* 27, $\quad({}^{r}\mathrm{A}_1^3\,\mathrm{A}_2)$
- $\mathrm{A}_1\,\mathrm{A}_2^2$ *if and only if* $\mathrm{DO}(F)$ *is in the algebraic set* $V(\mathrm{A}_1\,\mathrm{A}_2^2)$ *of an ideal* $(\mathrm{A}_1\,\mathrm{A}_2^2)$ *defined by 14 polynomials whose degree profile is* 12, 15^2, 18^5, 21^4, 24 *and* 27, $\quad(\mathrm{A}_1\,\mathrm{A}_2^2)$
- A_3, $\mathrm{A}_1\,\mathrm{A}_3$, $\mathrm{A}_2\,\mathrm{A}_3$, ${}^{r}\mathrm{A}_3^2$ *or* ${}^{r}\mathrm{A}_1^2\,\mathrm{A}_3$ *if and only if* $\mathrm{DO}(F)$ *is in the algebraic set* $V(\mathrm{A}_3)$ *of an ideal* (A_3) *defined by 11 polynomials whose degree profile is* 9, 12^2, 15^2, 18^3, 21^2 *and* 27. $\quad(\mathrm{A}_3)$

Strata of dimension 2.

Proposition 3.6. *Let C be a plane quartic given by a GIT-semi-stable ternary quartic F over an algebraically closed field $\overline{K}$ of characteristic 0 that is not in a stratum of dimension less than* *or equal to 1, then C is of singularity type*

- $\mathrm{A}_1^2\,\mathrm{A}_2$ *if and only if* $\mathrm{DO}(F)$ *is in the algebraic set* $V(\mathbf{A_1^2\,A_2})$ *of an ideal* $(\mathbf{A_1^2\,A_2})$ *defined by 13 polynomials whose degree profile is* 12, 15, 18^2, 21^2, 24^2, 27^3 *and* 30^2, $(\mathrm{A}_1^2\,\mathrm{A}_2)$
- A_2^2 *if and only if* $\mathrm{DO}(F)$ *is in the algebraic set* $V(\mathbf{A_2^2})$ *of an ideal* $(\mathbf{A_2^2})$ *defined by 13 polynomials whose degree profile is* 12, 15, 18^3, 21^3, 24^2, 27^2 *and* 30, (A_2^2)
- ${}^r\mathrm{A}_1^4$ *(line and cubic) if and only if* $\mathrm{DO}(F)$ *is in the algebraic set* $V({}^{\mathbf{r}}\mathbf{A_1^4}_{\mathbf{cub}})$ *of an ideal* $({}^{\mathbf{r}}\mathbf{A_1^4}_{\mathbf{cub}})$ *defined by 10 polynomials whose degree profile is* 6, 9, 12, 15^2, 18, 21^2, 27 *and* 36. $({}^{\mathrm{r}}\mathrm{A}_1^4{}_{\mathrm{cub}})$
- ${}^r\mathrm{A}_1^4$ *(two conics) if and only if* $\mathrm{DO}(F)$ *is in the algebraic set* $V({}^{\mathbf{r}}\mathbf{A_1^4}_{\mathbf{con}})$ *of an ideal* $({}^{\mathbf{r}}\mathbf{A_1^4}_{\mathbf{con}})$ *defined by 22 polynomials whose degree profile is* 15, 18^4, 21^5, 24^4, 27^4, 30^2, 33 *and* 36. $({}^{\mathrm{r}}\mathrm{A}_1^4{}_{\mathrm{con}})$

<u>*Strata of dimension 3*</u>.

Proposition 3.7. *Let C be a plane quartic given by a GIT-semi-stable ternary quartic F over an algebraically closed field $\overline{K}$ of characteristic 0 that is not in a stratum of dimension less than or equal to 2, then C is of singularity type*

- A_1^3 *if and only if* $\mathrm{DO}(F)$ *is in the algebraic set* $V(\mathbf{A_1^3})$ *of an ideal* $(\mathbf{A_1^3})$ *defined by 26 polynomials whose degree profile is* 24^2, 27^5, 30^7, 33^6, 36^5 *and* 39, (A_1^3)
- $\mathrm{A}_1\mathrm{A}_2$ *if and only if* $\mathrm{DO}(F)$ *is in the algebraic set* $V(\mathbf{A_1A_2})$ *of an ideal* $(\mathbf{A_1A_2})$ *defined by 9 polynomials whose degree profile is* 12, 15, 18^2, 21^2, 24, 27 *and* 45, $(\mathrm{A}_1\mathrm{A}_2)$
- ${}^r\mathrm{A}_1^3$ *if and only if* $\mathrm{DO}(F)$ *is in the algebraic set* $V({}^{\mathbf{r}}\mathbf{A_1^3})$ *of an ideal* $({}^{\mathbf{r}}\mathbf{A_1^3})$ *defined by 9 polynomials whose degree profile is* 6, 9, 12, 15^2, 18, 21^2 *and* 27. $({}^{\mathrm{r}}\mathrm{A}_1^3)$

<u>*Strata of dimension 4*</u>.

Proposition 3.8. *Let C be a plane quartic given by a GIT-semi-stable ternary quartic F over an algebraically closed field $\overline{K}$ of characteristic 0 that is not in a stratum of dimension less than or equal to 3, then C is of singularity type*

- A_2 *if and only if* $\mathrm{DO}(F)$ *is in the algebraic set* $V(\mathbf{A_2})$ *of an ideal* $(\mathbf{A_2})$ *defined by 8 polynomials whose degree profile is* 12, 15, 18^2, 21^2, 24 *and* 27, (A_2)
- A_1^2 *if and only if* $\mathrm{DO}(F)$ *is in the complement of* $V(\mathbf{A_2})$ *with respect to the algebraic set* $V(\mathbf{A_1^2})$ *of the ideal* $(\mathbf{A_1^2})$ *defined by 6 polynomials whose degree profile is* 30, 33, 36^2, 39 *and* 42 . (A_1^2)

<u>*Strata of dimension 5*</u>.

Proposition 3.9. *Let C be a plane quartic given by a GIT-semi-stable ternary quartic F over an algebraically closed field $\overline{K}$ of characteristic 0 that is not in a stratum of dimension less than or equal to 4, then C is of singularity type*

- A_1 *if and only if* $\mathrm{DO}(F)$ *is in the algebraic set* $V(\mathbf{A_1})$ *of the ideal* $(\mathbf{A_1}) = (I_{27})$. (A_1)

Remark 3.10. We may notice that, unlike the other singularity types, the ideal $(\mathbf{A_1^2})$ used in Prop. 3.8 is not sufficient to fully characterise quartics with at least 2 nodes, because $V(\mathbf{A_1^2})$ also contains the set $V(\mathbf{A_2})$. The reason why we have done this is chiefly a computational one: generators for the ideal corresponding to $\overline{V(\mathbf{A_1^2})\backslash V(\mathbf{A_2})}$ do not seem to be computable within a reasonable time, whilst it was feasible to find generators for $V(\mathbf{A_1^2})$.

Remark 3.11. Prop. 3.9 is also clearly true for any characteristic $p > 0$, even for $p = 2$ or 3, by scaling the expression of I_{27} known for $\mathbb{Q}$ to the right power of 2 or 3, ensuring that it is equal to the discriminant of F. Provided the Dixmier-Ohno invariants generate the invariant algebra, Prop. 3.4 could also be proved in characteristic $p > 7$. We believe that the same should hold for the other propositions, yielding Conj. 3.3. However, such an assertion is difficult to prove, as the proof in characteristic 0 is based on Gröbner basis calculations (see Sec. 4). Nonetheless, we invite the reader to compare with Rem. 1.1.

4. Proof of Theorem 3.2

Throughout this section, it is still assumed that the field of definition is an algebraically closed field $\overline{K}$ of characteristic zero.

We obtained the generators of the ideal given in Sec. 3 by process of evaluation and interpolation on randomly chosen Hui normal forms, in the manner of [LR12]. It is therefore difficult to derive a direct proof from this. Instead, the proofs we devise below do not depend on the way we obtained them. They consist mainly in checking that the only normal forms with Dixmier-Ohno invariants that satisfy stratum equations for a prescribed types are of this type.

In order to reduce the number of verifications to be done, we first prove that the inclusion tree of the algebraic sets defined by the ideals of Prop. 3.4 to Prop. 3.9 is almost the same as the specialisation tree of the singular quartics given in Fig. 2.1. This is the content of Lem. 4.1 in Sec. 4.1.

It is then relatively easy to check computationally that the Dixmier-Ohno invariants of Hui normal forms for a given type generically satisfy the stratum relations we propose for that type. The most difficult point is to further verify that if these invariants cancel the relations of a substratum, this necessarily implies conditions on the parameters of the normal form so that the forms they define have the expected singularities for that substratum. This can be done almost directly, as explained in Sec. 4.2.

The most difficult case is certainly that of Hui normal forms of type A_1; to show that such a quartic necessarily admits other singularities when its invariants verify the equations of a substratum, in particular the equations of the A_1^2 or A_2 strata. We take advantage of the fact that these normal forms, whatever their parameters, always have at least one singularity of exactly A_1 type (see Rem. 2.2). We therefore expect them to have one or more other singularities if their invariants satisfy the relations of a substratum. In the proof, we thus use a polynomial in the parameters of the A_1 family whose cancellation is equivalent to having at least two singularities. This is the content of Lem. 4.4 in Sec. 4.3. We treat the quartics of type A_1^2 and A_2 in the same way (Lem. 4.5 and Lem. 4.6).

4.1. Dixmier-Ohno stratum inclusion tree.

The following lemma shows that the ideals given in Sec. 3 define an inclusion tree that is almost the same as Fig. 2.1.

Lemma 4.1. *With the notations of Sec. 3, the algebraic sets defined by the 18 ideals*

dim. 0, $({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{6}})$, $(\mathrm{A}_{\mathbf{2}}^{\mathbf{3}})$, $({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}\,\mathrm{A}_{\mathbf{3}})$, $(\mathrm{A}_{\mathbf{4}})$,	*dim. 3,* $(\mathrm{A}_{\mathbf{1}}^{\mathbf{3}})$, $(\mathrm{A}_{\mathbf{1}}\mathrm{A}_{\mathbf{2}})$, $({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{3}})$
dim. 1, $({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{5}})$, $({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{3}}\,\mathrm{A}_{\mathbf{2}})$, $(\mathrm{A}_{\mathbf{1}}\,\mathrm{A}_{\mathbf{2}}^{\mathbf{2}})$, $(\mathrm{A}_{\mathbf{3}})$,	*dim. 4,* $(\mathrm{A}_{\mathbf{2}})$, $(\mathrm{A}_{\mathbf{1}}^{\mathbf{2}})$,
dim. 2, $(\mathrm{A}_{\mathbf{1}}^{\mathbf{2}}\,\mathrm{A}_{\mathbf{2}})$, $(\mathrm{A}_{\mathbf{2}}^{\mathbf{2}})$, $({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}\ \mathbf{cub}}^{\mathbf{4}})$, $({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}\ \mathbf{con}}^{\mathbf{4}})$,	*dim. 5,* $(\mathrm{A}_{\mathbf{1}})$.

define an inclusion tree faithful to the quartic singularity type specialisation tree, with the notable exception of $V(\mathrm{A}_{\mathbf{2}}) \subset V(\mathrm{A}_{\mathbf{1}}^{\mathbf{2}})$.

The proof consists in checking that these 18 algebraic sets intersect as expected, which amounts to computing radicals of sums of two ideals according to the usual correspondence between algebraic sets and polynomial ideal (*Hilbert's Nullstellensatz*),

$$\begin{array}{ccc} \text{ideals} & \longleftrightarrow & \text{algebraic sets}\,, \\ (\mathrm{A}) + (\mathrm{A}') & \longrightarrow & V(\,(\mathrm{A}) + (\mathrm{A}')\,) = V(\mathrm{A}) \cap V(\mathrm{A}')\,, \\ \mathrm{Id}(V \cap V') = \mathrm{Rad}(\mathrm{Id}(V) + \mathrm{Id}(V')) & \longleftarrow & V \cap V'\,. \end{array}$$

We proceed by increasing dimension. Given a dimension d, we check that the pairwise intersections of the algebraic sets of ideals of dimension d with the algebraic sets of ideals of dimension $d-1$, or other ideals of dimension d, decompose as illustrated in Fig. 2.1.

For instance, the two by two intersections of the algebraic sets of $({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{6}})$, $(\mathrm{A}_{\mathbf{2}}^{\mathbf{3}})$, $({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}\,\mathrm{A}_{\mathbf{3}})$ and $(\mathrm{A}_{\mathbf{4}})$ are all equal to the unstable locus $(O) = (I_3, I_6, \ldots, I_{27})$. If we now consider $({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{5}})$, we can similarly check that we have $V({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{5}}) \cap V(\mathrm{A}_{\mathbf{4}}) = V(O)$, $V({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{5}}) \cap V({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}\,\mathrm{A}_{\mathbf{3}}) = V({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}\,\mathrm{A}_{\mathbf{3}})$, $V({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{5}}) \cap V(\mathrm{A}_{\mathbf{2}}^{\mathbf{3}}) = V(O)$ and $V({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{5}}) \cap V({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{6}}) = V({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{6}})$. When the dimension increases, more complex situations can occur, such as for example

$$V(\mathrm{A}_{\mathbf{3}}) \cap V({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{5}}) = V({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}\,\mathrm{A}_{\mathbf{3}})\,, \text{ and } V(\mathrm{A}_{\mathbf{3}}) \cap V(\mathrm{A}_{\mathbf{1}}\,\mathrm{A}_{\mathbf{2}}^{\mathbf{2}}) = V(\mathrm{A}_{\mathbf{4}}) \cup V({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}\,\mathrm{A}_{\mathbf{3}})\,.$$

From a computational point of view, the sum of two ideals is obtained by joining generators, from which a basis can be extracted by a Gröbner basis calculation. It takes a handful of seconds to compute in MAGMA a Gröbner basis of those ideals, over $\mathbb{Q}$, for the graded reverse lexicographical (or "grevlex") order $I_3 < I_6 < \cdots < I_{27}$ with weights 3, …, 27.

Remark 4.2. Upon performing the same computations in positive characteristic $p > 0$, we have observed that, depending on p, some of the expected inclusions may not occur. For example, $V(\mathrm{A}_{\mathbf{3}}) \cap V(\mathrm{A}_{\mathbf{1}}\,\mathrm{A}_{\mathbf{2}}^{\mathbf{2}})$ is equal to $V(\mathrm{A}_{\mathbf{4}}) \cup V({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}\,\mathrm{A}_{\mathbf{3}})$ over $\mathbb{Q}$, while we only have $V(\mathrm{A}_{\mathbf{4}})$ modulo $p = 11$.

4.2. **Strata of small dimensions.** We first consider Prop. 3.4 to Prop. 3.7. We can already notice that it is immediate to check that a Hui normal form for those types A have their Dixmier-Ohno invariants in $V(\mathrm{A})$. The main task is therefore to show that there is no quartic with singularity type A whose invariants are in the algebraic set defined by one of the substrata below A in the specialisation graph on Fig. 2.1.

Since the dimensions are sufficiently small here, we can follow a direct approach. Suppose we start with a Hui normal form F_{A} for some type A such that $\mathrm{DO}(F_{\mathrm{A}}) \in V(\mathrm{A}') \subseteq V(\mathrm{A})$, where A' is a substratum which is more degenerate than A. Then we want to show that the singularity type of F_{A} is more degenerate than just A. We start by computing a decomposition into primary ideals of the ideal obtained from the generators of (A') that we evaluate in $\mathrm{DO}(F_{\mathrm{A}})$. We then reduce the equations which define the locus of the singular points of F_{A} modulo each component of this decomposition. By further Gröbner basis calculations, applied this time to each of these polynomial systems, we determine explicitly the singular points of F_{A}. It remains to quantify the type of each of them to finally ensure that the singularity type of forms F_{A} whose invariants are restricted to $\mathrm{V}(\mathrm{A}')$ cannot remain A.

Most of these computations can be easily done by the MAGMA commutative algebra engine, except the determination of the type of a singular point. This part relies on another native MAGMA package, developed by M. Harrison in full generality for schemes.

Example 4.3. As an illustration, we take the example of a normal form for A_1^3,

$$F_{\alpha,\beta,\gamma}(x,y,z) = (y^2 + \alpha\, yx + x^2)z^2 + (\beta\, y^2x + \gamma\, yx^2)z + y^2x^2\,,$$

that we specialise to $({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}\ \mathbf{con}}^{\mathbf{4}})$. The decomposition in primary ideals of $({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}\ \mathbf{con}}^{\mathbf{4}})$ evaluated at $\mathrm{DO}(F_{\alpha,\beta,\gamma})$ yields only one radical ideal,

$$\alpha^2 + \beta^2 + \gamma^2 - \alpha\,\beta\,\gamma - 4 \;=\; 0\,,$$

and modulo this ideal, the form $F_{\alpha,\beta,\gamma}$ has generically 4 singular points,

$$(1:0:0)\,,(0:1:0)\,,(0:0:1) \text{ and } \big((2\,\beta-\alpha\,\gamma)\,(\beta^2-4):(2\,\gamma-\alpha\,\beta)\,(\gamma^2-4):(\gamma^2-4)\,(\beta^2-4)\big)\,.$$

It remains to determine the singularity type of these points, and it turns out to be A_1. In other words, we proved that a normal form for A_1^3 specialised to $V({}^{\mathbf{r}}\mathbf{A}_{\mathbf{1}\ \mathbf{con}}^{\mathbf{4}})$ is generically of type A_1^4. Whether it is ${}^{r}\mathrm{A}_{1\ con}^{4}$ or ${}^{r}\mathrm{A}_{1\ cub}^{4}$ does not really matter here, the important point is that $F_{\alpha,\beta,\gamma}$ is no longer of type A_1^3.

Note that for particular values of the parameters, the fourth singular point can collide with one of the first three. But we can check that one of the singularities is then more complex than A_1. Typically, for $\beta=\gamma=0$, and thus $\alpha=\pm 2$, it turns out that $F_{\alpha,\beta,\gamma}$ is of type $\mathrm{A}_1^2\mathrm{A}_3$.

4.3. **Specialisations of Hui normal forms for A_1, A_1^2 and A_2.** The method described in Sec. 4.2 does not work for the strata A_1, A_1^2, and A_2, because the Gröbner basis computations needed for the method did not finish within a reasonable time. In this section, we discuss a different method that we used to prove Prop. 3.8 and 3.9, *i.e.* the correctness of the algorithm for the strata of dimensions 4 and 5.

We first notice that all normal forms for A_1 have more than one singularity, or not, depending on whether a certain polynomial in its parameters is zero, or not. In particular, because of the special form of these quartic equations, the node singularity cannot degenerate further. It yields this technical lemma.

Lemma 4.4. *Let*

$$F_{\mathrm{A}_1}(x,y,z)=yz^3+(\alpha\,y^2+x^2)z^2+(\beta\,y^3+\gamma\,y^2x+yx^2)z+\delta\,y^4+\epsilon\,y^3x$$

be a Hui normal form for A_1 over an algebraically closed field $\overline{K}$ of characteristic 0, then $F_{\mathrm{A}_1}(x,y,z)$ has at least two singularities if and only if α, β, γ, δ and ϵ are a zero of a polynomial $\iota_{\mathrm{A}_1}(\alpha,\beta,\gamma,\delta,\epsilon)$ of degree 6 in α, β and δ, degree 8 in ϵ, and degree 9 in γ (cf. [Bom+23a, file `G3SingularProof.m`] for its expression). In all cases, F_{A_1} has at least one A_1-singularity.

Proof. Let $(x:y:z)\in\mathbb{P}^2$ be a singular point of F_{A_1}. We thus have

$$\left\{\begin{array}{rcl} 0 &=& x^2\,y+2\,x^2\,z+\gamma\,x\,y^2+\beta\,y^3+2\,\alpha\,y^2\,z+3\,y\,z^2\,,\\ 0 &=& x^2\,z+3\,\epsilon\,x\,y^2+2\,\gamma\,x\,y\,z+4\,\delta\,y^3+3\,\beta\,y^2\,z+2\,\alpha\,y\,z^2+z^3\,,\\ 0 &=& 2\,x\,y\,z+2\,x\,z^2+\epsilon\,y^3+\gamma\,y^2\,z\,,\\ 0 &=& x^2\,y\,z+x^2\,z^2+\epsilon\,x\,y^3+\gamma\,x\,y^2\,z+\delta\,y^4+\beta\,y^3\,z+\alpha\,y^2\,z^2+y\,z^3\,. \end{array}\right.$$

Let us first assume that $y=0$, and thus $0=2\,x^2\,z=x^2\,z+z^3=2\,x\,z^2=x^2\,z^2$. As expected, it yields the singular point $(1:0:0)$ which is of type A_1 whatever the parameters α, β, γ, δ and ϵ are.

Now, assume that $y\neq 0$, *i.e.* without loss of generality $y=1$. By eliminating x and z in the system of equations thanks to a Gröbner basis computation, we arrive at a single equation, which corresponds to $\iota_{\mathrm{A}_1}(\alpha,\beta,\gamma,\delta,\epsilon)$. Restricted to parameters α, …, ϵ that are zeros of this polynomial, the form F_{A_1} has one or several singular points of the form $(x:1:z)$. Otherwise, the only singular point of F_{A_1} is $(1:0:0)$. ☐

Let us now return to our original problem, *i.e.* to show that there is no quartic with a unique singularity A_1 whose invariants are in the algebraic set defined by one of the ideals (A') of Sec. 3. So, let us evaluate the generators of those ideals into the Dixmier-Ohno invariants of the Hui normal form. The obtained polynomials define an ideal in $\mathbb{Q}[\alpha,\beta,\gamma,\delta,\epsilon]$ of which we can compute a Gröbner basis, this time for the *grevlex* order $\alpha < \epsilon < \gamma < \delta < \beta$. It remains then to reduce $\iota_{\mathrm{A}_1}(\alpha, \beta, \gamma, \delta, \epsilon)$ modulo this basis and to check that the result is indeed 0.

We made these calculations for the 17 ideals $(\mathrm{A}') = (\mathbf{A_1^2})$, $(\mathbf{A_2})$ ... and check modulo each of them that $\iota_{\mathrm{A}_1}(\alpha, \beta, \gamma, \delta, \epsilon)$ is indeed zero. In MAGMA this is a matter of seconds, except for a very small number of cases. The longest calculation is for $(\mathbf{A_1\,A_2^2})$, which takes half an hour on a standard laptop.

We can verify in the same way that a quartic of type A_1^2, or A_2, cannot have its Dixmier-Ohno invariants in the algebraic set of one of the ideals below $V(\mathbf{A_1^2})$, or $V(\mathbf{A_2})$. For this we need an equivalent of Lem. 4.4. These are the two technical lemmas that follow. We omit their proofs, because they are quite similar to that of Lem. 4.4.

Lemma 4.5. *Let $F(x, y, z) = (y^2 + x^2)z^2 + (\alpha\, yx^2 + \beta\, x^3)z + y^2x^2 + \gamma\, yx^3 + \delta\, x^4$ be a Hui normal form for A_1^2 over an algebraically closed field $\overline{K}$ of characteristic 0. Then $F(x, y, z)$ has at least three singularities if and only if α, β, γ and δ are a zero of a polynomial $\iota_{\mathrm{A}_1^2}(\alpha, \beta, \gamma, \delta)$ of degree 5 in δ, degree 6 in β and γ, and degree 8 in α (cf. [Bom+23a, file `G3SingularProof.m`] for its expression). In all cases, F has at least two A_1-singularities.*

Lemma 4.6. *Let $F(x, y, z) = yz^3 + (\alpha\, y^2 + \beta\, yx + x^2)z^2 + (\gamma\, y^3 + \delta\, y^2x)z + y^3x$ be a Hui normal form for A_2 over an algebraically closed field $\overline{K}$ of characteristic 0. Then $F(x, y, z)$ has at least two singularities if and only if α, β, γ and δ are a zero of a polynomial $\iota_{\mathrm{A}_2}(\alpha, \beta, \gamma, \delta)$ of degree 5 in α and γ, and degree 7 in β and δ (cf. [Bom+23a, file `G3SingularProof.m`] for its expression). In all cases, F has at least one A_2-singularity.*

Finally, once we check by an easy calculation that the Hui normal forms for A_1^2 and A_2 have their respective Dixmier-Ohno invariants in $V(\mathbf{A_1^2})$ and $V(\mathbf{A_2})$, we know for sure that Prop. 3.8 is valid.

5. From the singularity type to the reduction type

The Hui stratification of plane quartics has been considered in previous sections. Unfortunately, this stratification does not behave well under the action of (the group-scheme) SL_3. Indeed, if we were to try to construct a quotient space for $\overline{\mathcal{F}}_{3,4}^{\mathrm{Hui}}$ by the action of SL_3, this space would not satisfy the valuative criterion for properness [Stacks, Tag 03IX], as shown in Ex. 5.1. Nevertheless, we manage in Sec. 3 to partially relate the Hui stratification and the GIT compactification.

In this section, we aim to characterise the stable reduction type of a curve in terms of the singularities of GIT-semi-stable quartics, see Thm. 5.2. Using the results in Sec. 3, Corollary 5.5 relates then Dixmier-Ohno invariants and (Deligne-Mumford stable) reduction types.

Notation. Throughout this section, K is a non-archimedean local field with valuation v of residue characteristic $p \neq 2$, 3, 5, 7. Let R denote its valuation ring, π a uniformiser and k its residue field.

Example 5.1. Let us consider the smooth plane quartic $\pi^5 z^4 + x^2z^2 + \pi^2 y^2 z^2 + \pi y^3 z + y^4 + yx^3 = 0$. The $\mathrm{SL}_3(K)$-change of variables $(x : y : z) \mapsto (\pi^{-1}x : y : \pi z)$ yields the model $\pi z^4 + x^2z^2 + y^2z^2 + y^3z + y^4 + \pi^3 yx^3 = 0$. But these two K-isomorphic smooth plane quartics are not $\mathcal{O}_K$-isomorphic, because their special fibres are the the curves in Ex. 3.1, which have different singularity types.

5.1. **Stable reduction types and invariants.** In this section we relate the singularity type and the (stable) reduction type of a plane quartic. We refer to Sec. 2.3 for the notation for representing stable curves in Fig. 5.1.

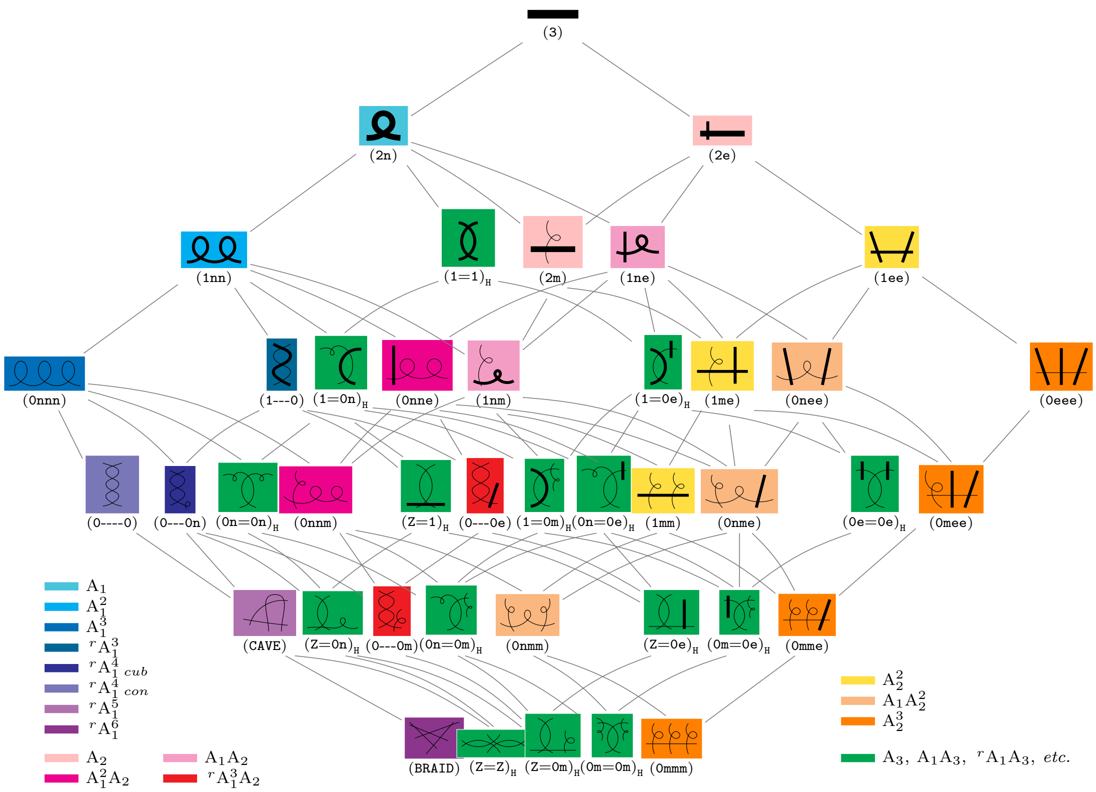

FIGURE 5.1. Stable model and singularity types of non-hyperelliptic genus 3 curves (the genus of a component is indicated by its thickness)

Theorem 5.2. *Let C/K be a plane quartic curve given by a smooth ternary quartic F over a non-archimedean local field K of residue characteristic $p \neq 2, 3, 5, 7$. Suppose that the reduction $\bar{F}$ of the ternary quartic is GIT-semi-stable. Then the stable reduction type of C is, in terms of the singularity type of $\bar{F}$, as in Tab. 5.1.*

Remark 5.3. The wildcard $*$ used in Tab. 5.1d and Tab. 5.1e represents any possible component in a reduction type, restricted to the set of 42 labels as defined exactly in Fig. 5.1. For example, $(\mathtt{1} = *)_{\mathtt{H}}$ can be any of the four types $(\mathtt{1} = \mathtt{1})_{\mathtt{H}}$, $(\mathtt{1} = \mathtt{0n})_{\mathtt{H}}$, $(\mathtt{1} = \mathtt{e})_{\mathtt{H}}$ and $(\mathtt{1} = \mathtt{0m})_{\mathtt{H}}$. But it cannot be $(\mathtt{Z} = \mathtt{1})_{\mathtt{H}}$, which is covered by the expression $(* = \mathtt{1})_{\mathtt{H}}$ instead.

Tab. 5.1 shows that curves with stable reduction type of the form $(* = *)_{\mathtt{H}}$ can have GIT-semi-stable special fibres of different singularity types. Ex. 5.1 and Ex. 3.1 illustrate this phenomenon for a quartic F with stable reduction type $(\mathtt{1} = \mathtt{0n})$. Linear changes of variables yield quartics with singularity types A_3 or A_1A_3, while the normalised invariants are the same for both types of singularities.

Our proof of this theorem is based on the Hui stratification, which as it stands is only valid in residue characteristic $p \neq 2, 3, 5, 7$ (*cf.* Rem. 2.3). But this limitation can be removed if we restrict ourselves to A_1 and Tab. 5.1a (*cf.* Prop. A.1). More generally, $p \neq 2$, 3 seems to be sufficient, but this might require revising the Hui stratification modulo 5 and 7, as we cannot preclude the existence of other singularity types.

Sing. type	Red. type
A_1	`(2n)`
A_1^2	`(1nn)`
A_1^3	`(0nnn)`
${}^rA_1^3$	`(1---0)`
${}^rA_{1\,con}^4$	`(0----0)`
${}^rA_{1\,cub}^4$	`(0---0n)`
${}^rA_1^5$	`(CAVE)`
${}^rA_1^6$	`(BRAID)`

(a) A_1-singularities

Sing. type	Red. type
Smooth	`(3)`
$\geq A_4$, incl. c^2	$(*)_{\mathtt{H}}$

(b) Other types

Sing. type	Red. type
A_2	`(2e)` or `(2m)`
A_1A_2	`(1ne)` or `(1nm)`
$A_1^2A_2$	`(0nne)` or `(0nnm)`
${}^rA_1^3A_2$	`(0---0e)` or `(0---0m)`
A_2^2	`(1ee)` or `(1em)` or `(1mm)`
$A_1A_2^2$	`(0nee)` or `(0nem)` or `(0nmm)`
A_2^3	`(0eee)` or `(0eem)` or `(0emm)` or `(0mmm)`

(c) A_2-singularities

Sing. type	Red. type
A_3	$(\mathtt{1} = *)_{\mathtt{H}}$
A_1A_3	$(\mathtt{0n} = *)_{\mathtt{H}}$ or $(* = \mathtt{0n})_{\mathtt{H}}$
${}^rA_1^2A_{3\,con}$	$(\mathtt{Z} = *)_{\mathtt{H}}$
A_2A_3	$(* = \mathtt{0e})_{\mathtt{H}}$ or $(* = \mathtt{0m})_{\mathtt{H}}$
${}^rA_3^2$	$(\mathtt{1} = *)_{\mathtt{H}}$ or $(\mathtt{0n} = *)_{\mathtt{H}}$ or $(\mathtt{Z} = *)_{\mathtt{H}}$

(d) A_3-singularities

Sing. type	Red. type
${}^rA_1A_3$	$(\mathtt{1} = *)_{\mathtt{H}}$ or $(* = \mathtt{1})_{\mathtt{H}}$
${}^rA_1^2A_{3\,cub}$	$(\mathtt{0n} = *)_{\mathtt{H}}$ or $(* = \mathtt{0n})_{\mathtt{H}}$
${}^rA_1^3A_3$	$(\mathtt{Z} = *)_{\mathtt{H}}$
${}^rA_1A_2A_3$	$(* = \mathtt{0e})_{\mathtt{H}}$ or $(* = \mathtt{0m})_{\mathtt{H}}$
${}^rA_1A_3^2$	$(\mathtt{1} = *)_{\mathtt{H}}$ or $(\mathtt{0n} = *)_{\mathtt{H}}$ or $(\mathtt{Z} = *)_{\mathtt{H}}$

(e) ${}^rA_3$-singularities

TABLE 5.1. From singularity to reduction types

Example 5.4. It is easy to produce smooth plane quartics whose normalised invariants modulo a uniformiser π are those of c^2 and with stable reduction type any prescribed type that can be obtained from a hyperelliptic curve. Indeed, let $f \in K[u]$ be a degree 8 polynomial with roots producing the desired stable reduction type (see [DDMM19]). Let $F(x, y, z)$ be a homogeneous degree 4 polynomial such that $f(u) = F(1, u, u^2)$. Such an F exists and is unique modulo $xz-y^2$. Then the genus 3 non-hyperelliptic curve defined by $t^2 = F(x, y, z)$ and $\pi^s t = xz - y^2$, with s large enough, has the desired stable reduction type too and admits a plane quartic model as $(xz - y^2)^2 - \pi^{2s}F(x, y, z) = 0$.

It is difficult to find plane quartics satisfying the assumption of Thm. 5.2 of having a GIT-semi-stable reduction, since the creation of such plane quartics often involves large degree field extensions. However, its mere existence is enough to obtain explicit criteria in terms of invariants. Indeed, by [LLLR21, Prop. 3.13], for each quartic F, there exists an equivalent GIT-semi-stable plane quartic model such that its invariants are minimal in the sense that their valuations are as small as possible among all quartics equivalent to F. The connection in Sec. 3 between invariants and singularity types will allow us to drop the GIT-semi-stable assumption from Thm. 5.2.

To explain this further, we need to introduce the following notation. Suppose that F is a smooth ternary quartic over K and that I_w is an invariant of weight w for the action of $\mathrm{SL}_3(\mathbb{C})$. We define the normalised valuation v_{S} with respect to a set of invariants S as in [LLLR21], *i.e.*

$$v_{\mathrm{S}}(I_w(F)) = v(I_w(F))/w - \min_{J_\omega \in \mathrm{S}} \{v(J_\omega(F))/\omega\}$$

(in this formula, J_ω denotes an invariant of weight ω). The two sets relevant here are the set DO of Dixmier-Ohno invariants or the set ι of Shioda invariants written in terms of Dixmier-Ohno invariants [LLLR21, Prop. 5.6]. We denote the corresponding normalised valuations by v_{DO}

and v_ι. We also say that F has normalised invariants if $v(I_w(F)) = v_{\mathrm{DO}}(I_w(F))$ for one (and therefore all) invariants of positive weight.

We know that the special fibre of the stable model of the curve $F=0$ is a smooth plane quartic if and only if $v_{\mathrm{DO}}(I_{27}(F)) = 0$ [LLLR21, Thm. 1.9]. When this is not the case, we still know that this special fibre is a smooth hyperelliptic curve of genus 3 if and only if $v_{\mathrm{DO}}(I_3(F)) = 0$ and $v_\iota(I_3(F)^5\, I_{27}(F)) = 0$ [LLLR21, Thm. 1.10].

We extend these definitions to a set of invariants (A), *i.e.* $v_{\mathrm{S}}((\mathrm{A})(F)) = \min_{J_w \in (\mathrm{A})} v_{\mathrm{S}}(J_w(F))$ Especially, the condition $v_{\mathrm{DO}}((\mathrm{A})(F)) > 0$ may be interpreted as the reduction of a plane quartic, with invariants the normalised ones of F, being in the algebraic set defined by the reduction of the ideal generated by (A).

We can now reformulate Thm. 5.2 in terms of Dixmier-Ohno invariants.

Corollary 5.5. *Let $C\colon F=0$ be a plane quartic curve over a non-archimedean local field K of residue characteristic $p \neq 2,3,5,7$. Then, under Conjecture 3.3, candidates for its reduction type are returned by Alg. 1 modified in such a way that:*

- *Membership tests "$(I_3 : I_6 : \cdots : I_{27}) \in V(\mathrm{A})$", are replaced by "$v_{\mathrm{DO}}\,(\mathrm{A})(F) \;>\; 0$";*
- *The outputs are converted according to Tab. 5.1.*

We note that the reduction modulo π of normalised invariants encodes exactly the reduction type for singularity types A_1 (*e.g.* Tab. 5.1a), offers very few possibilities for A_2 (*e.g.* Tab. 5.1c), and gives only the hint $(*=*)_{\mathtt{H}}$ for A_3 (*e.g.* Tab. 5.1d and Tab. 5.1e).

Example 5.6. Consider the Dixmier-Ohno invariants

$$\begin{aligned}(&\text{-}2^{-4}\,3^{-2},\ \text{-}2^{-12}\,3^{-6},\ \text{-}2^{-12}\,3^{-8}+O(\pi),\ 2^{-12}\,3^{-7}+O(\pi),\ \text{-}2^{-14}\,3^{-12},\ \text{-}2^{-17}\,3^{-10}+O(\pi),\ 2^{-22}\,3^{-15}+O(\pi),\\ &\text{-}2^{-22}\,3^{-12}+O(\pi),\ 2^{-24}\,3^{-17}+O(\pi),\ 2^{-24}\,3^{-15}+O(\pi),\ \text{-}2^{-29}\,3^{-18}+O(\pi),\ \ 2^{-32}\,3^{-16}\,7+O(\pi),\ 0)\,.\end{aligned}$$

They can be those of ternary quartics whose reduction modulo π is a Hui normal form with 6 nodes, for instance $F = x\,y\,z\,(z+y+x)+\pi\,z^4$. Moreover, these invariants are already normalised, as they all have non-negative valuation, and at least one of them has valuation 0 at π, as the residue characteristic was supposed to be greater than 7. Consider then the generators of the ideal $({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{6}})$,

$$\begin{aligned}\langle\, &2^4\,3^2\,I_6 + I_3^2,\ \ 3^2\,I_9 - I_3^3,\ \ 3\,J_9 + I_3^3,\ \ 3^4\,I_{12} + 4\,I_3^4,\ \ 2\,3^2\,J_{12} + I_3^4,\ \ 2^2\,3^5\,I_{15} + I_3^5,\\ &2^2\,3^2\,J_{15} - I_3^5,\ \ 3^5\,I_{18} - I_3^6,\ \ 3^3\,J_{18} - I_3^6,\ \ 2\,3^4\,I_{21} - I_3^7,\ \ 2^4\,3^2\,J_{21} + 7\,I_3^7,\ \ I_{27}\,\rangle\,.\end{aligned}$$

Generically, their normalised valuations with respect to the Dixmier-Ohno invariants $(I_3\,,\ldots I_{27})$ are $(\infty,1,1,\infty,1,1,1,1,1,1,1,\infty)$. Indeed, $2^4 3^2 I_6 + I_3^2$ is identically equal to zero, causing the first normalised valuation to be ∞, and $3^2 I_9 - I_3^3 = -2^{-12}3^{-6} + \mathcal{O}(\pi) + 2^{-12}3^{-6}$ generically has valuation 1, et cetera. In other words, $v_{\mathrm{DO}}\,({}^{\mathbf{r}}\mathrm{A}_{\mathbf{1}}^{\mathbf{6}})(F) = 1$, and we deduce from Cor. 5.5 and Alg. 1 that the reduction type is (`BRAID`) (corresponding to the purple label on Fig. 5.1).

5.2. **Proof of Theorem 5.2.** We split the proof of Theorem 5.2 into different cases according to only having singularities of type A_1, at worst A_2, at worst A_3 or at least an A_4 singularity.

5.2.1. *Nodal singularities.* Let $C\colon F=0$ be a plane quartic such that its reduction $\bar{F}=0$ modulo π defines a plane quartic with singularities at worst of type A_1, *i.e.* at any of the cases in Tab. 5.1a. In this case the plane quartic gives already a stable model for the curve and it is straightforward to compute the reduction type of the special fibre.

More precisely, we develop here for the reader's convenience the details for the A_1 case. The Hui classification (see Tab. 2.1) allows us to write

$$F = (z^2+yz)x^2 + (\gamma y^2 z + \epsilon y^3)x + (yz^3 + \alpha y^2 z^2 + \beta y^3 z + \delta y^4) + \pi\, G = 0\,,$$

and then the genus 2 curve is given by the hyperelliptic model

$$t^2 + (\gamma y^2 z + \epsilon y^3)t + (z^2 + yz)(yz^3 + \alpha y^2 z^2 + \beta y^3 z + \delta y^4) = 0\,.$$

This is a geometric genus 2 curve, further singularities would produce more singularities in $\bar{F}$. For example, following the same strategy, the A_1^2 case gives the arithmetic genus 2 curve

$$t^2 + x^2(\alpha y + \beta x)t + x^2(y^2 + x^2)(y^2 + \gamma yx + \delta x^2)$$

consisting of an elliptic curve with a node. This proves Thm. 5.2 for the cases in Tab. 5.1a.

Remark 5.7 (see Prop. A.1 in App. A)**.** In fact, we can give another proof for the converse of this result, of a more geometric nature. If one starts with a plane quartic whose stable reduction is of one of the types (`2n`), (`1nn`), (`0nnn`), (`1---0`), (`0nnn`), (`1---0`), (`0----0`), (`0----0n`), (`CAVE`) and (`BRAID`), then the canonical embedding of the stable model is an embedding of the stable genus 3 curve into $\mathbb{P}^3_R$ whose only singularities in the special fibre are of type A_1. This can be verified by doing the appropriate Riemann-Roch dimension computations on the dualising sheaf of the stable curve. This is done in the proof of Prop. A.1 for some but not all of the cases.

5.2.2. *Singularity type* A_2. We consider now quartics with at least a singularity of type A_2 and not worse singularities, that is the 7 cases in Tab. 5.1c.

Lemma 5.8. *We can place the* A_2 *singularity in such a way that the equation of the plane quartic is of the shape*

$$z^2x^2 + (\beta yz^2 + \delta y^2 z + y^3)x + (yz^3 + \alpha y^2 z^2 + \gamma y^3 z) + \pi^s\, G = 0\,. \tag{1}$$

Proof. In the cases of $\mathrm{A}_1\mathrm{A}_2$, ${}^r\mathrm{A}_1^3\mathrm{A}_2$, and A_2^2, this is relatively straightforward. In the cases of $\mathrm{A}_1^2\mathrm{A}_2, \mathrm{A}_1\mathrm{A}_2^2$, and A_2^3, we need to do a bit more work. We get a Hui model of the shape $(y^2 + \alpha yx + x^2)z^2 + (\beta y^2 x + \gamma yx^2)z + y^2x^2$, and for each α, β, and γ that is equal to -2 we have a cusp. If for example $\alpha = -2$, then the change of variables $(x, y, z) = (x', x' + y', z')$ will give a model in the shape of Eq. (1), up to permuting the variables. Note that we use here that $\beta + \gamma \neq 0$ in this case, as required in [Hui79]. □

Starting with an equation of the shape of Eq. (1), we can take $t = z^2x$ and making the reduction modulo π, we recover an arithmetic genus 2 curve piece given by

$$t^2 + (\beta yz^2 + \delta y^2 z + y^3)t + z^2(yz^3 + \alpha y^2 z^2 + \gamma y^3 z) = 0\,.$$

We also find another component, of arithmetic genus 1. Write Eq. (1) as

$$a_{400}\pi^{e_1}x^4 + (a_{310}\pi^{e_2}y + a_{301}\pi^{e_3}z)x^3 + (z^2 + y(a_{220}\pi^{e_4}y + a_{211}\pi^{e_5}z))x^2 +$$
$$(y^3 + \delta y^2 z + \beta yz^2 + a_{103}\pi^{e_6}z^3)x + (a_{004}\pi^{e_7}z^4 + yz^3 + \alpha y^2z^2 + \gamma y^3 z + a_{040}\pi^{e_8}y^4) = 0\,,$$

with the a_{ijk} equal to 0 or with $v(a_{ijk}) = 0$. We now multiply x by π^{-2s} and z by π^s where $s = \min\{e_1/6, e_2/4, e_3/3, e_4/2, e_5\}$. We multiply the equation by π^{2s} and we obtain

$$x(a_{400}\pi^{e_1-6s}x^3 + a_{310}\pi^{e_2-4s}yx^2 + a_{301}\pi^{e_3-3s}x^2z + xz^2 + a_{220}\pi^{e_4-2s}xy^2 + a_{211}\pi^{e_5-s}xyz + y^3) = O(\pi).$$

Under the inverse of this transformation, any point, except for the ones on the component $x = 0$, is mapped to the A_2 singularity at $(1:0:0)$ on the original Hui model. As in subsection 5.2.1, further singularities on the arithmetic genus 2 curve can be read from further singularities on the original plane quartic. The singularities on the arithmetic genus 1 component cannot be read from the singularities of the reduction of the plane quartic. The above changes of variables produce a construction of the different components of the stable reduction of the curve and the reduction type of the special fibre is easily computable in each case.

Remark 5.9. A generalised version of Lem. A.1 in the Appendix of [KLS20] implies the components of positive arithmetic genus should appear in the special fibre of any stable model and are not contracted. To check that this generalised version of the lemma holds, we should check this for nodal curves of geometric genus 0 and positive arithmetic genus, *e.g.* a genus 0 curve T with one node which intersects the rest of the special fibre at least once. The proof is similar to the proof in [KLS20] and is done by careful bookkeeping. The idea is to make a regular model. Following Lem. 3.21 of [Liu02], the node turns into a chain of $\mathbb{P}^1$'s and these $\mathbb{P}^1$'s get subsequently contracted when we are making the stable model. So there will still be a geometric genus 0 component in the special fibre with one self-intersection and at least one more intersection with the rest of the stable model.

Note that in the case of multiple A_2-singularities, the different components of arithmetic genus 1 do not intersect in the special fibre, as they are contracted to their distinct respective cusps on the Hui model. Therefore, we get that each A_2-singularity gives rise to a different "genus 1 tail", *i.e.* a component of arithmetic genus 1, intersecting the rest of the stable reduction in exactly 1 point. This proves Thm. 5.2 for the cases in Tab. 5.1c.

5.2.3. *Singularity type* A_3. Next, we consider the cases in Tab. 5.1d and Tab. 5.1e with at least a singularity of type A_3 and no worse singularities. According to Tab. 2.1, we can place the A_3 singularity in such a way that the equation of the plane quartic is of the shape

$$F = a_{004}\pi^{e_1}z^4 + (a_{103}\pi^{e_2}x + a_{013}\pi^{e_3}y)z^3 + (x^2 + a_{112}\pi^{e_4}xy + a_{022}\pi^{e_5}y^2)z^2 + (\alpha y^2x + a_{031}\pi^{e_6}y^3)z + y^4 + \beta y^3x + \gamma y^2x^2 + yx^3 + x^2z\,G = 0\,,$$

where G is a degree 1 polynomial in x and y with coefficients of positive valuation. Consider the new variables $(X, Y, Z) = (x, \pi^s y, \pi^{2s}z)$ where $s = \min\{e_1/4, e_2/2, e_3/3, e_4, e_5/2, e_6\}$. This gives an embedding of $C = \{F = 0\} \subseteq \mathbb{P}^2 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 : (x : y : z) \mapsto ((x : y : z), (X : Y : Z))$.

Modulo π, and if $x \neq 0$, we get the arithmetic genus 1 component

$$\{((1 : y : z), (1 : 0 : 0)) : z^2 + \alpha y^2z + y^4 + \beta y^3 + \gamma y^2 + y\}\,.$$

Notice that the homogenisation of this equation is a singular model of the arithmetic genus 1 curve mentioned above and that the point at infinity actually corresponds to two points. If $Z \neq 0$, modulo π we obtain the points $((0 : 0 : 1), (X : Y : 1))$ where X and Y satisfy the equation

$$X^2 + \alpha XY^2 + Y^4 + a_{031}\pi^{e_6-s}Y^3 + a_{022}\pi^{e_5-2s}Y^2 + a_{112}\pi^{e_4-s}XY + a_{013}\pi^{e_3-3s}Y + a_{103}\pi^{e_2-2s}X + a_{004}\pi^{e_1-4s} = 0\,.$$

This is again an arithmetic genus 1 curve, whose projective closure is again a singular model. The point at infinity corresponds to two points. These two arithmetic genus 1 components intersect at the double point $((0 : 0 : 1), (1 : 0 : 0))$.

If no more singularities appear on any of the arithmetic genus 1 curves we obtain reduction type $(\mathtt{1} = \mathtt{1})_{\mathtt{H}}$ for the quartic. From the original quartic equation we can only control the singularities appearing in one of the arithmetic genus 1 components. We do not have any information about the possible singularities of the second component. A case by case inspection gives the reduction type possibilities in Tab. 5.1d.

For the cases in Tab. 5.1e, we proceed as follows: we study first the cases ${}^r\mathrm{A}_1\mathrm{A}_3$ and ${}^r\mathrm{A}_1^2\mathrm{A}_3$ and then their degenerations. In the first case we can write the quartic as

$$x(y^2z + x(\alpha z^2 + \beta yz + xz + y^2)) + \pi^s\,G = 0\,.$$

The naive reduction of this model gives an arithmetic genus 1 curve. The change of variables $X = x$, $Y = \pi^{s/4}$ and $Z = \pi^{s/2}z$ gives the quartic $Z(XY^2 + \alpha X^2Z + gZ^3) = 0$, where g is the coefficient of z^4 in G, defining another arithmetic genus 1 curve. The map $C \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ given by $(x:y:z) \mapsto ((x:y:z),(X:Y:Z))$ produces two arithmetic genus 1 curves intersecting at two points. That is, a degeneration of $(\mathtt{1=1})_{\mathtt{H}}$. Further singularities on the first arithmetic genus 1 component can be read from further singularities of the original plane quartic. A priori we cannot see further singularities on the second arithmetic genus 1 component.

For the last case, namely ${}^{r}\mathrm{A}_1^2\mathrm{A}_3$, which is very degenerated, we change the strategy and we use Tim Dokchitser's machinery [Dok21] to compute regular models. This strategy is especially suitable for this case, but this is also an opportunity to show how it could have been used to prove the previous cases in a different way. The valuation polytope associated to a plane curve of the shape

$$(xz-y^2)(xz-\alpha y^2)+\pi^s\,G=0\,,$$

corresponding to a plane quartic with a ${}^{r}\mathrm{A}_1^2\mathrm{A}_3$ singularity type has an edge (after switching y and z and making $z=1$ as in [Bom+23b, Fig. 7.2a]) joining the points corresponding to the monomials 1, xy and x^2y^2. From [Bom+23b, Fig. 7.3], we see that we can (only) obtain any of the degeneration types $(*=*)_{\mathtt{H}}$ by playing with the valuations of the coefficients of G.

All of this together proves Thm. 5.2 for the cases in Tab. 5.1d and 5.1e.

5.2.4. *Singularity type* A_4. If a plane quartic has an A_4 singularity, we can assume, according to Hui, that it is given by an equation of the form

$$x^2z^2+2\,y^2xz+y^4+\alpha\,y^3x+\beta\,y^2x^2+yx^3+\pi\,G=0\,.$$

After multiplying x by a π^s, dividing z by π^s with $4s<1$ and renaming, we obtain a quartic of the form $(xz+y^2)^2+\pi\,G=0$, with singularity type c^2. We introduce the new variable $t=(xz+y^2)/\pi^{1/2}$ to get the equation $t^2+G=0$. Modulo π this produces a hyperelliptic equation $\bar{t}^2+\bar{G}(1,\bar{s},-\bar{s}^2)=0$ and hence, a hyperelliptic reduction type $(*)_{\mathtt{H}}$. The exact reduction type is encoded in the "cluster picture" of $G(1,s,-s^2)$ as explained in [DDMM19]. This proves Thm. 5.2 for the cases in Tab. 5.1b.

5.3. **Applications to databases.** We applied Alg. 1 to a dataset consisting of more than 82 000 curves by Sutherland [Sut19]. Our method only works for primes $p>7$ and there are 137 496 pairs (C,p) for which $p>7$ and p is a prime of bad reduction of a plane quartic C appearing in the dataset. For 131 673 of them, the reduction has only one A_1 singularity, and the stable reduction type is `(2n)`. Other common types were A_1^2 (`(1nn)`, 3511 cases) and A_2 (`(2e)` or `(2n)`, 1829 cases). Not all the singularity types occurred. For example, there was no curve with three cusps in the dataset. The total computation took a little less than 1650 seconds or about 10 milliseconds per case on a machine with an AMD EPYC 7713 CPU. For the computation, which was done with MAGMA version 2.28-8, only 74 MB of memory was needed. The largest part of the computation time is spent on evaluating the equations for the A_1^3 stratum on the Dixmier-Ohno invariants of the curves.

Appendix A. Node-only singularity quartics

Thm. 5.2 reports eight stable reduction types whose characterisation with singularities is one-to-one, they correspond to quartics with only node singularities. The following result gives an independent and more geometric proof of this.

Proposition A.1. *Let $C\colon F=0$ be a smooth plane quartic over a non-archimedean local field K. Suppose that the reduction $\bar{F}$ of the ternary quartic is GIT-semi-stable. Then the stable reduction type of the plane quartic C is* `(2n)`, *respectively* `(1nn)`, `(0nnn)`, `(1---0)`, `(0nnn)`, `(1---0)`, `(0----0)`, `(0----0n)`, `(CAVE)` *and* `(BRAID)`, *if and only if the singularity type of the reduction $\bar{F}$ is* A_1, *respectively* A_1^2, A_1^3, ${}^{r}\mathrm{A}_1^3$, ${}^{r}\mathrm{A}_{1\,con}^4$, ${}^{r}\mathrm{A}_{1\,cub}^4$, ${}^{r}\mathrm{A}_1^5$ *and* ${}^{r}\mathrm{A}_1^6$.

Proof. For ternary quartics F such that $\bar{F}$ is GIT-semi-stable with A_1^k singularities, reducible or not, we know that there exists a plane quartic model whose special fibre only has nodes and therefore is semi-stable (it is actually stable). Moreover, we immediately know the stable reduction type.

Conversely, let us start with a curve whose stable reduction type is one of the eight stated, and which we assume to be non-hyperelliptic. Since we have to relate it to certain quartic singularities, we must first embed its special fibre in $\mathbb{P}^2$. This can be done using techniques of a geometric nature, which depend on the type considered. We do this for the two extremal types, `(2n)` and `(BRAID)`, the other types can be done very similarly.

For `(2n)`, we are looking at a curve C of genus 2 with two points R and S glued to each other. We want to show that the canonical sheaf embeds it into $\mathbb{P}^2$. The canonical sheaf is obtained by taking $K_{\widetilde{C}} + R + S$ and considering the differentials for which the sum of the residues at R and S is zero. We know that $h^0_{\widetilde{C}}(K_{\widetilde{C}} + R + S) = 3$ and $h^0_{\widetilde{C}}(K_{\widetilde{C}} + R) = h^0_{\widetilde{C}}(K_{\widetilde{C}} + S) = h_{\widetilde{C}}(K_{\widetilde{C}}) = 2$ by Riemann-Roch. The sum of the residues of a meromorphic differential is 0, so the functions will automatically satisfy the residue condition. By Riemann-Roch $h^0_{\widetilde{C}}(K_{\widetilde{C}} + R + S - P) = 2$ for all P, so there will be a well-defined map to $\mathbb{P}^2$. The embedding condition is not satisfied at the points R and S, but that makes sense, as they are mapped to the same point. For any other two points P and Q, we claim that $h_{\widetilde{C}}(K_{\widetilde{C}} + R + S - P - Q) = 1$. It suffices to check that $P + Q \not\sim R + S$. As there is only one degree 2 map from $\widetilde{C}$ to $\mathbb{P}^1$, the hyperelliptic involution, the only way in which $P + Q \sim R + S$ can hold is if both P and Q and R and S are hyperelliptic involutions of each other. As we were looking at non-hyperelliptic reduction, R and S are not hyperelliptic involutions of each other. Therefore, we get an embedding outside of R and S. To see that the image in $\mathbb{P}^2$ is a nodal curve, we note that there exists a section vanishing twice at R, but only once at S. This follows from the fact that $h_{\widetilde{C}}(K_{\widetilde{C}} - S) = 1 > 0 = h_{\widetilde{C}}(K_{\widetilde{C}} - S - R)$.

For `(BRAID)`, we look at four copies of $\mathbb{P}^1$ glued in total at six pairs of points. Assume that the glued points are the disctinct points 0, 1, and α, extending the field if necessary. For each $\mathbb{P}^1$, we can take $\operatorname{div}(dx)$ as canonical divisor, which will give $-2\,\infty$. The canonical sheaf is now obtained by putting gluing conditions on four-tuples of differential forms of the shape $\frac{ax+b}{x(x-1)(x-\alpha)}dx$. This will give rise to 6 gluing conditions on an 8-dimensional vector space, but the sum of all the gluing conditions will be 0 as the sum of the residues on any $\mathbb{P}^1$ is 0. So there are at most 5 gluing conditions, and it is easy to check by linear algebra that there are really 5 linear independent conditions so that we get a potential map to $\mathbb{P}^2$. In this case, it is possible to explicitly compute the image of the map, see that it is four lines and check that it is an embedding, similarly to the `(2n)` case. □

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