COMPUTATION OF FOURIER COEFFICIENTS OF AUTOMORPHIC FORMS OF TYPE G_2

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ABSTRACT. In a recent work, we found formulas for the Fourier coefficients of automorphic forms of type G_2: holomorphic Siegel modular forms on Sp_6 that are theta lifts from G_2^c, and cuspidal quaternionic modular forms on split G_2. We have implemented these formulas in the mathematical software SAGE. In this paper, we explain the formulas of our recent paper and the SAGE implementation. We also deduce some theoretical consequences of our SAGE computations.

CONTENTS

1. INTRODUCTION

The purpose of this paper is to give computations of Fourier coefficients of automorphic forms of type G_2. The automorphic forms we compute come in two flavors. First, there are vector-valued holomorphic Siegel modular forms on Sp_6, that are exceptional theta lifts from algebraic modular forms for the group G_2^c. Here G_2^c is a group of type G_2 that is split at every finite place and for which $G_2^c(\mathbf{R})$ is compact. The second sort of automorphic forms we work with are quaternionic modular forms on *split* G_2. The forms we compute arise as exceptional theta lifts from algebraic modular forms for the group F_4^c, which is a group of type F_4 that is split at every finite place and for which $F_4^c(\mathbf{R})$ is compact. In fact, quaternionic modular forms on split G_2 have a *weight*, which is a positive integer. In [Pol23] it is proved that every cuspidal quaternionic modular form of even weight at least 6 arises as an exceptional theta lift from F_4^c, so not much generality is lost by computing with these theta lifts.[1]

That it is possible to compute exactly the Fourier coefficients of these exceptional theta lifts is a consequence of the results of [Pol23]. In an accompanying SAGE [Sag22] file [Pol24], we have implemented the formulas of [Pol23] in the case where all the data that goes into the theta lift is unramified. Consequently, in this case, the Fourier coefficients of the lift are exactly determinable on a computer. The resulting computer calculations[2] are not only satisfying–for example, they give a partial simultaneous check on the formulas of [Dal23] and [Pol23]–but they also have theoretical consequences.

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[1]It is also very reasonable to suspect that every cuspidal quaternionic modular form of odd weight at least 5 is an exceptional theta lift from F_4^c, but we have not proved this. The quaternionic modular forms of weights $1, 2, 3$ do not, however, arise as theta lifts from F_4^c.

[2]Our calculations were done on a Macbook Pro with an Apple M2 chip and 24 GB of RAM, running Sagemath version 9.7.

We now say a bit more about some theoretical consequences of these calculations, deferring a discussion of the results of [Pol23] and the SAGE computations to later sections.

1.1. Siegel modular forms of genus three.

Let Sp_{2n} denote the symplectic group, consisting of matrices g with $g \left(\begin{smallmatrix} 0_n & 1_n \\ -1_n & 0_n \end{smallmatrix} \right) g^t = \left(\begin{smallmatrix} 0_n & 1_n \\ -1_n & 0_n \end{smallmatrix} \right)$. Holomorphic Siegel modular forms are a certain type of automorphic forms for this group. Explicitly, set S_n the $n \times n$ symmetric matrices, and denote by $\mathcal{H}_n = \{Z = X + iY : X, Y \in S_n(\mathbf{R}), Y > 0\}$ the so-called Siegel upper half-space of degree n. Here $Y > 0$ means that Y is positive-definite. The group $\mathrm{Sp}_{2n}(\mathbf{R})$ acts on $\mathcal{H}_n$ via the formulas $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) Z = (aZ + b)(cZ + d)^{-1}$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{Sp}_{2n}(\mathbf{R})$, and $Z \in \mathcal{H}_n$, set $J(g, Z) = cZ + d$, which lives in $\mathrm{GL}_n(\mathbf{C})$. Let (ρ, V) be a finite-dimensional algebraic representation of $\mathrm{GL}_n(\mathbf{C})$. If $\Gamma \subseteq \mathrm{Sp}_{2n}(\mathbf{Z})$ is a congruence subgroup, a holomorphic Siegel modular form of weight ρ and level Γ is a holomorphic function $f : \mathcal{H}_n \to V$ satisfying $f(\gamma Z) = \rho(J(\gamma, Z))f(Z)$ for all $\gamma \in \Gamma$; one also imposes a moderate growth condition.

If $\Gamma = \mathrm{Sp}_{2n}(\mathbf{Z})$, we say that f has level one. Holomorphic Siegel modular forms have a Fourier expansion, which we explicate in the level one case. Denote $S_n(\mathbf{Z})^\vee$ the half-integral symmetric $n \times n$ matrices. I.e., $T \in S_n(\mathbf{Q})$ is in $S_n(\mathbf{Z})^\vee$ if the diagonal entries of T are integers, and the off-diagonal entries are integers divided by 2. If f is a level one Siegel modular form of weight (ρ, V), then one can write $f(Z) = \sum_{T \in S_n(\mathbf{Z})^\vee : T \geq 0} a_f(T) e^{2\pi i (T, Z)}$. Here $T \geq 0$ means that T is positive semi-definite, $(T, Z) = \mathrm{tr}(TZ) \in \mathbf{C}$, and $a_f(T) \in V$. The vectors $a_f(T)$ are called the Fourier coefficients of f. It is known [Ibu02] that, given ρ, there exists an explicitly determinable finite set C_ρ of half-integral symmetric matrices, so that if f is a level one Siegel modular form of weight ρ and $a_f(T) = 0$ for all $T \in C_\rho$, then $f = 0$. Consequently, computing finitely many Fourier coefficients of a level one Siegel modular form completely determines it.

The dual group of Sp_6 is $\mathrm{SO}_7(\mathbf{C})$, which receives a map from $G_2(\mathbf{C})$, the dual group of G_2. Langlands functoriality thus predicts a lifting of automorphic representations on groups of type G_2 to automorphic representations of Sp_6. One of the results of [Pol23], combined with work of Gross-Savin [GS98], Magaard-Savin [MS97], and Gan-Savin [GS23], allows one a way to computationally and provably produce instances of this lift. We have implemented this computation in SAGE, and one consequence of the SAGE computations is the following theorem.

To set up the theorem, we need some results of Chenevier-Taibi [CT20]. In [CT20], the authors have computed the dimension of the space of level one holomorphic Siegel modular forms of various weights. The following table is built from their computations.

(k_1, k_2)	$\lambda = (k_1 + 2k_2 + 4, k_1 + k_2 + 4, k_2 + 4)$	$m(\lambda)$
$(0, 4)$	$(12, 8, 8)$	1
$(2, 4)$	$(14, 10, 8)$	1
$(3, 3)$	$(13, 10, 7)$	1
$(0, 6)$	$(16, 10, 10)$	2
$(3, 4)$	$(15, 11, 8)$	1
$(6, 2)$	$(14, 12, 6)$	1
$(5, 3)$	$(15, 12, 7)$	1
$(7, 2)$	$(15, 13, 6)$	1
$(9, 1)$	$(15, 14, 5)$	1
$(6, 3)$	$(16, 13, 7)$	2
$(8, 2)$	$(16, 14, 6)$	2

In the table, $m(\lambda)$ denotes the dimension of the space of level one cuspidal Siegel modular forms for Sp_6 of weight λ. The parameter (k_1, k_2) has to do with algebraic modular forms on G_2^c, and will be explained in section 3.

Theorem 1.1. *If λ is in the above table, then every level one cuspidal eigenform of weight λ has all Satake parameters in $G_2(\mathbf{C})$.*

1.2. **Quaternionic modular forms on split G_2.** While the paper [Pol23] discusses the holomorphic Siegel modular forms of genus three, its main results concern the (quaternionic) modular forms on split G_2. The split group G_2 of this Dynkin type does not have a symmetric space $G_2(\mathbf{R})/K$ with a $G_2(\mathbf{R})$-invariant complex structure. Thus, there is no notion of holomorphic modular forms on this group.

A great replacement for the holomorphic modular forms was found by Gross-Wallach [GW96] and Gan-Gross-Savin [GGS02]. To briefly describe these objects, note that the maximal compact subgroup K of $G_2(\mathbf{R})$ is $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mu_2$, where the first $\mathrm{SU}(2)$ is the so-called long root $\mathrm{SU}(2)$, and the second is the short root $\mathrm{SU}(2)$. Forgetting the second $\mathrm{SU}(2)$ factor, one has a surjection $K \to \mathrm{SU}(2)/\mu_2 \simeq \mathrm{SO}(3)$. For a positive integer ℓ, let $\mathbf{V}_\ell$ be the irreducible complex representation of K that is the pull-back of the $(2\ell + 1)$-dimensional irreducible representation of $\mathrm{SO}(3)$.

If $\Gamma \subseteq G_2(\mathbf{R})$ is a congruence subgroup, a level Γ quaternionic modular form on G_2 of weight ℓ is a smooth function $\varphi : \Gamma \backslash G_2(\mathbf{R}) \to \mathbf{V}_\ell$ of moderate growth satisfying

(1) $\varphi(gk) = k^{-1}\varphi(g)$ for all $k \in K$;
(2) $\mathcal{D}_\ell \varphi \equiv 0$, for a certain linear, first-order differential operator $\mathcal{D}_\ell$.

The modular form φ is cuspidal if and only if φ is bounded. We say φ is of level one if $\Gamma = G_2(\mathbf{Z})$.

The quaternionic modular forms of weight ℓ have a Fourier expansion, similar in spirit to the Fourier expansion of holomorphic Siegel modular forms. We explicate this Fourier expansion in the level one case: For every real binary cubic $f(u, v) = au^3 + bu^2v + cuv^2 + dv^3$, there is a completely explicit moderate growth function (defined in terms of K-Bessel functions) $W_{f,\ell} : G_2(\mathbf{R}) \to \mathbf{V}_\ell$ satisfying properties (1), (2) of the definition of quaternionic modular forms of weight ℓ. The function $W_{f,\ell}(g)$ is 0 if the discrimant of the cubic f is negative. And, if φ is a level one cuspidal quaternionic modular form of weight ℓ, then $\varphi_Z(g) = \sum_{f \text{ integral}} a_\varphi(f) W_{f,\ell}(g)$. Here $\varphi_Z(g)$ is a certain compact integral transform of φ, which uniquely determines it, and the $a_\varphi(f)$ are complex numbers, called the Fourier coefficients of φ.

While *a priori* the Fourier coefficients of cuspidal quaternionic modular forms could be transcendental numbers, the main result of [Pol23] is that if $\ell \geq 6$ is even, then there is a basis of the space of level Γ, weight ℓ cuspidal quaternionic modular forms whose Fourier coefficients all lie in the cyclotomic extension of $\mathbf{Q}$. Moreover, the proof is constructive, giving the exact Fourier expansion of such G_2-cusp forms in an explicitly computable way. More specifically, we prove that every cuspidal quaternionic modular form of even weight $\ell \geq 6$ arises as a theta lift from the group F_4^c, and also explicitly compute the Fourier coefficients of such theta lifts, whether the weight is even or odd. We have implemented these explicit formulas of [Pol23] in the case when all the data that goes into the theta lift is unramified. The result is the ability to compute, on a computer, finitely many Fourier coefficients of some level one cuspidal quaternionic modular forms on G_2.

Associated to every integral binary cubic form f is a cubic ring S_f, i.e., a commutative ring which is rank three as a $\mathbf{Z}$-module; see [GGS02]. If f_1, f_2 are in the same $\mathrm{GL}_2(\mathbf{Z})$ orbit, then S_{f_1} is isomorphic to S_{f_2}. It follows immediately from the proof of this correspondence that, associated to f is in fact also an orientation of the rank two $\mathbf{Z}$-module $S_f/\mathbf{Z}$, by which we mean a generator of $\wedge^2(S_f/\mathbf{Z})$. If f_1 and f_2 are in the same $\mathrm{SL}_2(\mathbf{Z})$-orbit, then S_{f_1} is isomorphic to S_{f_2} as oriented cubic rings. Now, it is an easy consequence of the existence of the Fourier expansion of quaternionic modular forms, that if $f_1 = g \cdot f_2 = \det(g)^{-1} f_2((u, v)g)$ for $g \in \mathrm{GL}_2(\mathbf{Z})$, then $a_\varphi(f_1) = \det(g)^\ell a_\varphi(f_2)$. Here the integer ℓ is the weight of φ. Thus, if S is an oriented cubic ring, we can write $a_\varphi(S)$ for the associated Fourier coefficient.

Here is one theoretical consequence of our ability to compute finitely many Fourier coefficients of some cuspidal level one quaternionic modular forms on G_2. In a recent paper, Dalal [Dal23] has given an explicit formula for the dimension of the space $S_\ell(G_2(\mathbf{Z}))$ of level one cuspidal quaternionic modular forms of weight $\ell \geq 3$. It follows from his formulas that $S_9(G_2(\mathbf{Z}))$ and $S_{11}(G_2(\mathbf{Z}))$ are each one-dimensional.

Theorem 1.2. *The one-dimensional spaces $S_9(G_2(\mathbf{Z}))$ and $S_{11}(G_2(\mathbf{Z}))$ contain nonzero elements $F_9 \in S_9(G_2(\mathbf{Z}))$ and $F_{11} \in S_{11}(G_2(\mathbf{Z}))$ for which all Fourier coefficients are integers. Moreover, if B is an order in a quadratic étale algebra, then the Fourier coefficient $a_{F_9}(\mathbf{Z} \times B) = 0$ and $a_{F_{11}}(\mathbf{Z} \times B) = 0$.*

The latter part of Theorem 1.2 gives some evidence for a conjecture of Gross [Li], as we now explain. For every level one cuspidal holomorphic modular eigenform h on PGL_2 of weight $2k$, the Arthur multiplicity conjecture predicts the existence of a cuspidal lift F_h to G_2, which is a level one cuspidal quaternionic modular eigenform of weight k. Suppose now that E is a totally real etale cubic algebra, with maximal order $\mathcal{O}_E$. Gross has suggested that the square Fourier coefficient $a_{F_h}(\mathcal{O}_E)^2$ should be related to the central L-value $L(h \otimes V_E, 1/2)$, where V_E is the two-dimensional motive attached to E, i.e., where $\zeta_E(s) = \zeta(s)L(V_E, s)$. Suppose now that $E = \mathbf{Q} \times F$ with F real quadratic. Then $L(h \otimes V_E, s)$ factors as $L(h, s)L(h \otimes \epsilon_F, s)$ for a quadratic character ϵ_F. Hence if $L(h, 1/2) = 0$, then $L(h \otimes V_E, 1/2) = 0$. But when k is odd, the central value $L(h, 1/2)$ is indeed 0, so Gross' conjecture would predict $a_{F_h}(\mathbf{Z} \times B) = 0$ for such h. As the space of level one holomorphic cusp forms for PGL_2 is one-dimensional in weights 18 and 22, and so are the spaces $S_9(G_2(\mathbf{Z}))$ and $S_{11}(G_2(\mathbf{Z}))$, the forms F_9 and F_{11} of Theorem 1.2 are eigenforms, and should be lifts from PGL_2 as predicted by the Arthur multiplicity conjecture. Thus Theorem 1.2 gives some evidence for this conjecture of Gross.

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2. Exceptional algebra

We explain in this section a little bit of exceptional algebra, both "theoretically" and "computationally". The results in later sections depend upon this algebra.

2.1. **Octonions.** We begin by describing the octonions $\mathbb{O}$ with positive-definite norm form. These are an 8-dimensional $\mathbf{Q}$-vector space, with a law of composition $\mathbb{O} \times \mathbb{O} \to \mathbb{O}$ that is bilinear, but neither commutative nor associative. There is an element 1 with $1 \cdot x = x \cdot 1 = x$ for all $x \in \mathbb{O}$. Moreover, there is a positive definite quadratic form $n_{\mathbb{O}} : \mathbb{O} \to \mathbf{Q}$ that satisfies $n_{\mathbb{O}}(xy) = n_{\mathbb{O}}(x)n_{\mathbb{O}}(y)$ for all $x, y \in \mathbb{O}$. Let $(\ ,\ )$ denote the symmetric bilinear form induced by $n_{\mathbb{O}}$, so that $(x, y) = n_{\mathbb{O}}(x+y) - n_{\mathbb{O}}(x) - n_{\mathbb{O}}(y)$ for all $x, y \in \mathbb{O}$. Let $V_7 \subseteq \mathbb{O}$ denote the orthogonal complement of $\mathbf{Q} \cdot 1$. Define an involution, $*$, on $\mathbb{O}$ so that $1^* = 1$ and $x^* = -x$ if $x \in V_7$. Then $xx^* = x^*x = n_{\mathbb{O}}(x)$ for all $x \in \mathbb{O}$; $(xy)^* = y^*x^*$; and $x + x^* = \mathrm{tr}_{\mathbb{O}}(x)1$ for an element $\mathrm{tr}_{\mathbb{O}}(x) \in \mathbf{Q}$. For $z_1, z_2, z_3 \in \mathbb{O}$, define $(z_1, z_2, z_3) = \mathrm{tr}_{\mathbb{O}}(z_1(z_2z_3))$. It turns out that this quantity is equal to $\mathrm{tr}_{\mathbb{O}}((z_1z_2)z_3)$, and that one has $(z_1, z_2, z_3) = (z_2, z_3, z_1) = (z_3, z_1, z_2)$.

One way to create $\mathbb{O}$ is using quaternion algebras, as follows. Let H be a quaternion $\mathbf{Q}$-algebra, arbitrary subject to being ramified at infinity. Let $\gamma \in \mathbf{Q}^{\times}$ be negative. Define $\mathbb{O} = H \oplus H$, with multiplication $(x_1, y_1)(x_2, y_2) = (x_1x_2 + \gamma y_2^*y_1, y_2x_1 + y_1x_2^*)$. The norm is $n_{\mathbb{O}}((x, y)) = n_H(x) - \gamma n_H(y)$, the trace is $\mathrm{tr}_{\mathbb{O}}((x, y)) = \mathrm{tr}_H(x)$, and the involution is $(x, y)^* = (x^*, -y)$. Varying H and γ, it turns out, gives isomorphic data $(\mathbb{O}, n_{\mathbb{O}}, 1)$, which is why we have dropped H and γ from the notation of $\mathbb{O}$. From now on, we take H to be Hamilton's quaternions, i.e., the unique quaternion $\mathbf{Q}$-algebra ramified at 2 and ∞, with basis $\{1, i, j, k\}$ satisfying $i^2 = j^2 = k^2 = -1$ and $ij = k$. We take $\gamma = -1$.

There is a maximal $\mathbf{Z}$-order in $\mathbb{O}$, called Coxter's ring of integral octonions [Cox46]. We denote this ring by R. To construct it, set $e = (0, 1)$ and $h = \frac{1}{2}(i + j + k + e)$. Then, the following are a $\mathbf{Z}$

basis of R: $jh, e, -h, j, ih, 1, eh, ke$. These are the simple roots of the E_8 root lattice, with jh the extended node, 1 the branch vertex, and $e, -h, j, ih, 1, eh, ke$ going along longways.

The group G_2^c is defined as the algebraic **Q**-group of linear automorphisms of $\mathbb{O}$ that fixes 1 and respects the multiplication. Its Lie algebra can be identified with the kernel of the map $\wedge^2 V_7 \to V_7$ given by $x \wedge y \mapsto xy - yx$; see [Pol21b] for an explicit basis. An element of $\wedge^2 V_7$ acts on V_7 via the formula $(x \wedge y)(z) = (y, z)x - (x, z)y$; this gives the Lie algebra action of $\mathfrak{g}_2 = \mathrm{Lie}(G_2^c)$ on V_7.

Let $K = \mathbf{Q}(\sqrt{-1})$. Base-changing the group G_2^c to K, it becomes split. It is helpful to have a different basis of $\mathbb{O} \otimes K$, in which it is easy to write down nilpotent elements of $\mathfrak{g}_2 \otimes K$. We define

- $e_2 = \frac{1}{2}((0,1) - \sqrt{-1}(0,i))$
- $e_3^* = \frac{1}{2}((0,j) - \sqrt{-1}(0,k))$
- $e_3 = \frac{1}{2}((0,-j) - \sqrt{-1}(0,k))$.
- $e_2^* = \frac{1}{2}((0,-1) - \sqrt{-1}(0,i))$
- $\epsilon_1 = \frac{1}{2}((1,0) - \sqrt{-1}(i,0))$
- $\epsilon_2 = \frac{1}{2}((1,0) + \sqrt{-1}(i,0))$
- $e_1 = \frac{1}{2}((j,0) - \sqrt{-1}(k,0))$
- $e_1^* = \frac{1}{2}((-j,0) - \sqrt{-1}(k,0))$

This basis realizes $\mathbb{O} \otimes K$ as a split quadratic space: All the above basis elements are isotropic, and one has $(\epsilon_1, \epsilon_2) = 1$, $(e_i, e_j^*) = -\delta_{ij}$, and $(\epsilon_i, e_j) = (\epsilon_i, e_j^*) = 0$ for all i, j.

One can define elements of $G_2^c(K)$ by exponentiating nilpotent Lie algebra elements. For SAGE computation below, we will use the following nilpotent elements.

Suppose $u, v \in V_7 \otimes K$ satisfy $u^2 = uv = vu = v^2 = 0$; such a pair is said to be *null*. If moreover u, v span a two-dimensional subspace of V_7, then the stabilizer of the flag $\{0\} \subseteq Ku \subseteq \mathrm{Span}(u, v)$ is a Borel subgroup of $G_{2,K}^c$.

Lemma 2.1. *The pair e_3^*, e_1 is null. Let B be the Borel of $G_{2,K}^c$ stabilizing the flag $Ke_3^* \subseteq Ke_3^* + Ke_1$. Then the nilpotent $\bar{\mathfrak{n}}$ in $\mathfrak{g}_2 \otimes K$ opposite to B is spanned by the following six elements:*

(1) $\ell_1 = e_1^* \wedge e_2$,
(2) $\ell_2 = (\epsilon_1 - \epsilon_2) \wedge e_2^* + e_3 \wedge e_1$
(3) $\ell_3 = [\ell_1, \ell_2]$
(4) $\ell_4 = [\ell_3, \ell_2]/2$
(5) $\ell_5 = [\ell_4, \ell_2]/3$
(6) $\ell_6 = [\ell_5, \ell_1]$.

Proof. This follows directly from [Pol21b, Section 2.2]. $\square$

2.2. **The exceptional cubic norm structure.** We now describe a bigger exceptional algebraic structure. Namely, let $J = H_3(\mathbb{O})$ be the 27-dimensional **Q**-vector space consisting of elements

$$X = \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix}$$ with $c_1, c_2, c_3 \in \mathbf{Q}$ and $x_1, x_2, x_3 \in \mathbb{O}$. This J is called the exceptional cubic norm structure. The space J comes equipped with a cubic norm $N_J : J \to \mathbf{Q}$ defined as $N_J(X) = c_1 c_2 c_3 - c_1 n_{\mathbb{O}}(x_1) - c_2 n_{\mathbb{O}}(x_2) - c_3 n_{\mathbb{O}}(x_3) + (x_1, x_2, x_3)$.

Let $(\,,\,,\,)_J$ be the unique symmetric trilinear form on J satisfying $(X, X, X)_J = 6N_J(X)$ for all $X \in J$. For $X \in J$, let $X^\# \in J^\vee$ be the linear map given by $(Z, X^\#) = \frac{1}{2}(Z, X, X)$ for all $Z \in J$. One says that X has rank one if $X \neq 0$ but $X^\# = 0$. For $X, Y \in J$, one sets $X \times Y = (X + Y)^\# - X^\# - Y^\#$, then one has $(Z, X \times Y) = (Z, X, Y)$ for all $Z \in J$. If $U \in J$ satisfies $N_J(U) = 1$, one defines a symmetric bilinear form on J as $(X, Y)_U = (X, U^\#)(Y, U^\#) - (U, X, Y)$.

We will write elements X of J as $X = [c_1, c_2, c_3; x_1, x_2, x_3]$. Set $I = [1, 1, 1; 0, 0, 0]$, which has $N_J(I) = 1$. Then if $X = [c_1, c_2, c_3; x_1, x_2, x_3]$ and $X' = [c_1', c_2', c_3'; x_1', x_2', x_3']$, then

(1) $\qquad (X, X')_I = c_1 c_1' + c_2 c_2' + c_3 c_3' + (x_1, x_1')_{\mathbb{O}} + (x_2, x_2')_{\mathbb{O}} + (x_3, x_3')_{\mathbb{O}}.$

Observe that this bilinear form is positive-definite. It induces an identification of J with $J^\vee$, whose inverse we denote $\iota : J^\vee \to J$. If X, X' are as above, then

$$\iota(X^\#) = [c_2 c_3 - n(x_1), c_3 c_1 - n(x_2), c_1 c_2 - n(x_3); (x_2 x_3)^* - c_1 x_1, (x_3 x_1)^* - c_2 x_2, (x_1 x_2)^* - c_3 x_3]$$
$$\iota(X \times X') = [c_2 c_3' + c_2' c_3 - (x_1, x_1'), c_3 c_1' + c_3' c_1 - (x_2, x_2'), c_1' c_2 + c_1 c_2' - (x_3, x_3');$$
$$(x_2' x_3 + x_2 x_3')^* - c_1' x_1 - c_1 x_1', (x_3' x_1 + x_3 x_1')^* - c_2' x_2 - c_2 x_2', (x_1' x_2 + x_1 x_2')^* - c_3' x_3 - c_3 x_3'].$$

Set $J_R \subseteq J$ to be the set consisting of those $X = [c_1, c_2, c_3; x_1, x_2, x_3]$ with $c_i \in \mathbf{Z}$ and $x_i \in R$. This is an integral lattice in J. Following [EG96], we distinguish two different quadratic forms on this lattice. The first is $(\,,\,)_I$. For the second, define $\beta = \frac{1}{2}(-1 + i + j + k, 1 + i + j + k)$. Then $\beta^2 + \beta + 2 = 0$, so that $\mathrm{tr}_\mathbb{O}(\beta) = -1$ and $n_\mathbb{O}(\beta) = 2$. One sets $E = [2, 2, 2; \beta, \beta, \beta]$, so that $N_J(E) = 1$. The second quadratic form on J_R is $(\,,\,)_E$. This form is again positive-definite [EG96]. For an explicit expression for $(\,,\,)_E$, observe that we have $(X, X)_E = (E^\#, X)^2 - 2(E, X^\#)$, and the individual terms in this expression are computed in Lemma 4.6.

Set M_J^1 to be the algebraic $\mathbf{Q}$-group of linear automorphisms of J preserving the cubic norm. It is a simply-connected group of type E_6. There exists $\delta \in M_J^1(\mathbf{Q})$ for which $\delta E = I$. For SAGE computations, we will use an explicit choice of δ. To set up the result, if $\gamma \in J^\vee$ and $x \in J$, let $\Phi'_{\gamma,x} : J \to J$ be defined as $\Phi'_{\gamma,x}(z) = -\gamma \times (x \times z) + (\gamma, z)x + \frac{1}{3}(\gamma, x)z$. Then $\Phi'_{\gamma,x} \in \mathrm{Lie}(M_J^1)$ [PWZ19, Proposition 1.1], [Pol20, Section 3.3].

For $x, y, z \in \mathbb{O}$, set $V(x, y, z) = [0, 0, 0; x, y, z]$ and $V_1(x) = V(x, 0, 0)$, $V_2(y) = V(0, y, 0)$ and $V_3(z) = V(0, 0, z)$. Let $e_{11} = [1, 0, 0; 0, 0, 0]$ and similarly define $e_{22} = [0, 1, 0; 0, 0, 0], e_{33} = [0, 0, 1; 0, 0, 0]$.

Lemma 2.2. *For $\delta \in M_J^1(\mathbf{Q})$, one can take*

$$\delta = \exp(\Phi'_{e_{22},V_1(-1)}) \exp(\Phi'_{V_1(3/2),e_{22}}) \exp(\Phi'_{e_{22},V_1(-1)}) \exp(\Phi'_{V_1(1),e_{22}})$$
$$\times \exp(\Phi'_{e_{22},V_1(-\beta/2)}) \exp(\Phi'_{e_{11},V_2(-(\beta+1)/2)}) \exp(\Phi'_{V_3(-\beta/2),e_{11}}).$$

The $\Phi'_{\gamma,x}$ appearing in this product satisfy $(\Phi'_{\gamma,x})^3 = 0$.

Proof. The fact that the $(\Phi'_{\gamma,x})^3 = 0$ above follows from [PWZ19, Proposition 1.1]. That the above δ satisfies $\delta E = I$ can be verified by explicit computation in SAGE. $\square$

Remark 2.3. The δ of Lemma 2.2 was found by partial guess and check. One first finds a product of unipotent elements in $\mathrm{SL}_3(\mathbf{Q}(\beta))$ that move E to I. Then, with the aid of [Pol21a, page 23, after Proposition 3.0.9], one can express these unipotent elements in terms of the $\Phi'_{\gamma,x}$.

Let F_4^c be the algebraic subgroup of M_J^1 of elements that also fix I. We define J^0 to be the subspace of elements $X \in J$ with $c_1 + c_2 + c_3 = 0$; equivalently, J^0 is the orthogonal complement to I under the bilinear pairing $(\,,\,)_I$. It is preserved by F_4^c. There is a surjective F_4^c-equivariant map $\wedge^2 J^0 \to \mathfrak{f}_4$ from $\wedge^2 J^0$ to the adjoint representation of F_4^c. It is given by

$$X \wedge Y \mapsto \Phi_{X \wedge Y} := \Phi'_{\iota^{-1}(X),Y} - \Phi'_{\iota^{-1}(Y),X}.$$

Let V_1 be the kernel of this map. It is an irreducible representation of F_4^c of dimension 273.

We will require special vectors $X_!, Y_!$ in $J \otimes K$. To define them, let t be the square-root of -1 in K, to distinguish it from $i \in H$ and $i_K \in H_K$. Then, in $\mathbb{O}_K$, we define the elements

- $r_1 = \frac{1}{2}(0, 1 - t \cdot i_K)$
- $r_2 = r_1$
- $r_3 = -t(i_K, 0)$
- $s_1 = -t(0, i_K)$
- $s_2 = \frac{1}{2}(0, 1 + t \cdot i_K)$
- $s_3 = -\frac{1}{2}(1 + t \cdot i_K, 0)$.

Now, we set $X_! = [1, -1, 0, r_1, r_2, r_3]$ and $Y_! = [0, -1, 1, s_1, s_2, s_3]$ as elements of $J \otimes K$.

Lemma 2.4. *The wedge $X_! \wedge Y_! \in \wedge^2 J_K^0$ sits in V_1, and is a highest weight vector in V_1 for a Borel subgroup of $F_{4,K}^c$.*

Proof. This is proved in [Pol23, Lemma 3.1.3 and Example 3.1.4].   □

We will also need other vectors in J_K, obtained from $X_!$ and $Y_!$ by $F_4^c(K)$-automorphisms. To produce many such automorphisms, we use the following lemma. Recall that $e_{11} = [1, 0, 0; 0, 0, 0]$, $e_{22} = [0, 1, 0; 0, 0, 0]$, and $e_{33} = [0, 0, 1; 0, 0, 0]$.

Lemma 2.5. *Suppose $v' \in \mathbb{O} \otimes K$ is in the set $\{\epsilon_1, \epsilon_2, e_1, e_2, e_3, e_1^*, e_2^*, e_3^*\}$. Then $\Phi'_{u \wedge v}$, with $u = e_{11}, v = V(0, v', 0)$, $u = e_{11}, v = V(0, 0, v')$, and $u = e_{22}, v = V(v', 0, 0)$, are nilpotent elements in the Lie algebra $\mathfrak{f}_4$. They satisfy $(\Phi'_{u \wedge v})^3 = 0$.*

Proof. For some context regarding these Lie algebra elements, see [Pol21a, page 23]. One can verify in SAGE that the $(\Phi'_{u \wedge v})^3 = 0$.   □

2.3. SAGE implementation of octonions.

SAGE already has quaternion algebras implemented. Using the Cayley-Dickson construction, i.e., realizing $\mathbb{O}$ as $H \oplus H$ with H equal to Hamilton's quaterions, one can realize the octonions in SAGE. Namely, an octonion in SAGE is represented as a vector consisting of two elements of the quaternion algebra H, or two elements of the quaternion algebra $H \otimes K$. Our file, `g2_motives.sage`, contains functions to multiply two octonions, take their trace and conjugate, and compute the inner product of two octonions. Also included is the trilinear form $(x_1, x_2, x_3)_{\mathbb{O}}$.

To do computations, we have some specific bases of $\mathbb{O}$ and of $\mathbb{O} \otimes K$ hard-coded into the program. The first basis is the ordered Coxeter basis, which is a **Z**-basis of Coxeter's ring R. The basis is ordered as before: $[jh, e, -h, j, ih, 1, eh, ke]$. The second basis is the split basis of $\mathbb{O} \otimes K$, ordered as $[\epsilon_1, e_1, e_2, e_3, e_1^*, e_2^*, e_3^*, \epsilon_2]$. There is some built-in code to change basis from one to another, and to go from octonions, to vectors of length 8 of elements of **Q** or K. The Gram matrices for the trace pairing and the norm pairing, with respect to the Coxeter basis, are hard-coded into the file.

2.4. SAGE implementation of the exceptional cubic norm structure.

Building upon the octonions above, we represent elements of J or $J \otimes K$ as a list of length 6, $X = [c_1, c_2, c_3, x_1, x_2, x_3]$. Here the $c_i \in \mathbf{Q}$ or K and the $x_i \in \mathbb{O}$ or $\mathbb{O} \otimes K$ are octonions. The code contains functions to compute the norm of an element X, the element $\iota(X^{\#})$ or $\iota(X \times X')$, the pairings $(X, X')_I$ and $(X, X')_E$, and the trilinear form $(X_1, X_2, X_3)_J$. The Gram matrix for the pairing $(X, X)_E$, for $X \in J_R$, is hard-coded into the file `g2_motives.sage`.

3. SIEGEL MODULAR FORMS OF GENUS THREE

We begin with a discussion of results of [Pol23] that apply to Siegel modular forms of genus three. We then discuss the SAGE implementation of the Fourier coefficient formula.

3.1. Fourier expansion of the holomorphic exceptional theta lift.

In order to put Theorem 1.1 into context and to explain how it is proved, recall that this result asserts the existence of Siegel modular forms of genus three with all Satake parameters in $G_2(\mathbf{C})$. The Siegel modular forms arise as an exceptional theta lift from algebraic modular forms for the group G_2^c, as studied in Gross-Savin [GS98]. Specifically, there is a group H of type E_7, which is split at every finite place and whose real rank is 3. The group H supports a very special automorphic representation $\Pi_{\min,H}$ on $H(\mathbf{A})$ called the minimal automorphic representation, whose study was begun by Kim [Kim93]. Gross-Savin use automorphic functions in $\Pi_{\min,H}$ to lift automorphic forms on G_2^c to automorphic

forms on Sp_6, using the fact that $\mathrm{Sp}_6 \times G_2^c \to H$. Namely, if α is an automorphic form on $G_2^c(\mathbf{A})$ and $\Theta_f \in \Pi_{\min,H}$, then one can define the lift

$$\Theta_f(\alpha)(g) = \int_{G_2^c(\mathbf{Q}) \backslash G_2^c(\mathbf{A})} \Theta_f(gh) \alpha(h) \, dh$$

which is an automorphic form on Sp_6. It is proved in [GS98] that the automorphic forms $\Theta_f(\alpha)$ correspond to vectors in the $(\mathfrak{sp}_6, U(3))$-module generated by a vector-valued Siegel modular form.

Gross-Savin [GS98], Magaard-Savin [MS97] and Gan-Savin [GS23] have produced numerous and deep results on this exceptional theta lift. One question left open by these works, however, is to determine when $\Theta_f(\alpha)$ is nonzero for explicit α. One of the first main results of [Pol23] solves this question, in the case of level one: It gives explicit formulas to determine the Fourier coefficients of the Siegel modular form corresponding to $\Theta_f(\alpha)$.

Before describing this result of [Pol23], we review some prior work on the theta lift from G_2^c to Sp_6. Let W_3 denote the standard representation of GL_3, with basis w_1, w_2, w_3, and let $V_3 = \wedge^2 W_3$ the exterior square representation, with basis $v_1 = w_2 \wedge w_3, v_2 = w_3 \wedge w_1, v_3 = w_1 \wedge w_2$. Every irreducible algebraic representation V of $\mathrm{GL}_3(\mathbf{C})$ sits as the highest weight submodule in $Sym^{k_1}(V_3) \otimes Sym^{k_2}(W_3) \otimes \det(W_3)^{k_3}$ for integers k_1, k_2, k_3 with $k_1, k_2 \geq 0$. The highest weight of such a V is $(k_1 + k_2 + k_3, k_1 + k_3, k_3)$. Thus, if f is a level one Siegel modular form of weight (ρ, V), then the Fourier coefficients of f are naturally polynomials in $v_1, v_2, v_3, w_1, w_2, w_3$ of bi-degree (k_1, k_2).

For non-negative integers k_1, k_2, let $\lambda(k_1, k_2) = (k_1 + 2k_2 + 4, k_1 + k_2 + 4, k_2 + 4)$. For an irreducible representation ρ of $\mathrm{GL}_3(\mathbf{C})$, let $M_\rho^\Theta(\mathrm{Sp}_6(\mathbf{Z}))$ denote that space of level $\mathrm{Sp}_6(\mathbf{Z})$, weight ρ holomorphic Siegel modular forms (not necessarily cuspidal) which arise as (vector-valued) theta lifts $\Theta_f(\alpha)$. It is proved in [GS98] that for this space to be nonzero, the highest weight of ρ must be of the form $\lambda(k_1, k_2)$ for some non-negative integers k_1, k_2. If the highest weight of ρ is $\lambda(k_1, k_2)$, we abuse notation and denote this space also by $M_{k_1,k_2}^\Theta(\mathrm{Sp}_6(\mathbf{Z}))$.

The following result can be extracted from work of Gross-Savin, Magaard-Savin, and Gan-Savin. See [Pol23] for details.

Theorem 3.1. *One has the following results concerning the space $M_{k_1,k_2}^\Theta(\mathrm{Sp}_6(\mathbf{Z}))$.*

    (1) *(Gross-Savin) If $k_2 > 0$, the space $M_{k_1,k_2}^\Theta(\mathrm{Sp}_6(\mathbf{Z}))$ is contained in the space of cusp forms.*

    (2) *(Gross-Savin, Magaard-Savin, Gan-Savin) Suppose $F \in M_{k_1,k_2}^\Theta(\mathrm{Sp}_6(\mathbf{Z}))$ is a cuspidal Hecke eigenform. Then all Satake parameters of F lie in $G_2(\mathbf{C}) \subseteq \mathrm{SO}_7(\mathbf{C})$.*

While powerful, Theorem 3.1 leaves open the actual calculation of the elements of $M_{k_1,k_2}^\Theta(\mathrm{Sp}_6(\mathbf{Z}))$. In particular, for fixed k_1, k_2, it was not clear how to determine if $M_{k_1,k_2}^\Theta(\mathrm{Sp}_6(\mathbf{Z}))$ is nonzero. Results of [Pol23] (in theory), and of this present work (in practice) resolve these questions, at least when $k_2 > 0$.

We now explicate some portion of the results of [Pol23], referring to [Pol23] for more details. Recall that we denote by $\mathbb{O}$ the octonion algebra over $\mathbf{Q}$ with positive-definite norm form and $R \subseteq \mathbb{O}$ is Coxeter's order of integral octonions. We write $J = H_3(\mathbb{O})$ the exceptional cubic norm structure, which is the 27-dimensional $\mathbf{Q}$-vector space consisting of 3×3 Hermitian matrices with "coefficients" in $\mathbb{O}$ and let J_R be the integral lattice in J consisting of elements whose diagonal entries are in $\mathbf{Z}$ and off-diagonal entries are in R. Recall we denote by V_7 the elements of $\mathbb{O}$ with 0 trace and $K = \mathbf{Q}(\sqrt{-1})$. For $T \in J$ and $Z \in S_3(\mathbf{C})$ the 3×3 complex symmetric matrices, define $(T, Z)_I \in \mathbf{C}$ from equation (1), by viewing $S_3(\mathbf{Q}) \subseteq H_3(\mathbb{O})$ and extending the pairing $(,)_I$ by linearity from $\mathbf{Q}$ to $\mathbf{C}$. Concretely, if $T = [c_1, c_2, c_3, x_1, x_2, x_3]$ and $Z = [z_{11}, z_{22}, z_{33}, z_{23}, z_{31}, z_{12}]$ with $z_{ij} \in \mathbf{C}$ then

$$(T, Z)_I = c_1 z_{11} + c_2 z_{22} + c_3 z_{33} + \mathrm{tr}_{\mathbb{O}}(x_1) z_{23} + \mathrm{tr}_{\mathbb{O}}(x_2) z_{31} + \mathrm{tr}_{\mathbb{O}}(x_3) z_{12}.$$

Elements of J have a notion of *rank*, which is an integer in $\{0, 1, 2, 3\}$.

Suppose $u, v \in V_7 \otimes K$ satisfy $u^2 = uv = vu = v^2 = 0$; recall that such a pair is said to be *null*. Let $T \in J$ have off-diagonal entries $x_1, x_2, x_3 \in \mathbb{O}$. For non-negative integers k_1, k_2, set

$$P_{k_1,k_2}(T; u, v) = ((x_1, u)v_1 + (x_2, u)v_2 + (x_3, u)v_3)^{k_1}$$
$$\times ((x_1 \wedge x_2, u \wedge v)w_3 + (x_2 \wedge x_3, u \wedge v)w_1 + (x_3 \wedge x_1, u \wedge v)w_2)^{k_2}.$$

Here $(x, u) = \operatorname{tr}(x^*u)$ is the bilinear form associated to the norm on $\mathbb{O}$, and $(x \wedge y, u \wedge v) = (x, u)(y, v) - (x, v)(y, u)$. Moreover, in case either k_1 or k_2 is equal to 0, r^k is defined to be 1 if $k = 0$, regardless of if $r = 0$. Fortunately, this is the convention that appears to be used by SAGE.

Finally, so long as at least one of k_1, k_2 is positive, set

$$(2) \qquad F_{k_1,k_2}^{u,v}(Z) = \sum_{T \in J_R, \operatorname{rank}(T)=1} \sigma_3(d_T) P_{k_1,k_2}(T; u, v) e^{2\pi i (T,Z)_I}$$

where d_T is the largest integer for which $d_T^{-1} T \in J_R$.

For $T \in J$, $T = [c_1, c_2, c_3, x_1, x_2, x_3]$, set $T_0 = [c_1, c_2, c_3, \frac{1}{2}\operatorname{tr}_{\mathbb{O}}(x_1), \frac{1}{2}\operatorname{tr}_{\mathbb{O}}(x_2), \frac{1}{2}\operatorname{tr}_{\mathbb{O}}(x_3)]$, so that T_0 is a symmetric matrix, half-integral if $T \in J_R$. Then if $Z \in S_3(\mathbf{C})$, $(T, Z)_I = (T_0, Z)$, the latter the usual trace pairing on symmetric matrices. Consequently, the expression (2) for $F_{k_1,k_2}^{u,v}$ can be seen as a Fourier expansion of the holomorphic function $F_{k_1,k_2}^{u,v}$ on the Siegel upper half space of degree three.

Theorem 3.2 (See [Pol23]). *Let the notation be as above, with at least one of k_1, k_2 positive. Let $V \subseteq Sym^{k_1}(V_3) \otimes Sym^{k_2}(W_3) \otimes \det(W_3)^{k_2+4}$ be the highest weight submodule so that the highest weight of V is $(k_1 + 2k_2 + 4, k_1 + k_2 + 4, k_2 + 4)$. Then $F_{k_1,k_2}^{u,v}(Z)$ is a level one holomorphic Siegel modular form on Sp_6 of weight (ρ, V). Moreover, if $k_2 > 0$, then the $F_{k_1,k_2}^{u,v}(Z)$ span $M_{k_1,k_2}^{\Theta}(\operatorname{Sp}_6(\mathbf{Z}))$ as u, v vary over null pairs.*

Thus, combining Theorem 3.2 with Theorem 3.1 gives an explicit way of constructing cuspidal eigenforms on Sp_6, all of whose Satake parameters are in $G_2(\mathbf{C})$. The code implemented produces finitely many Fourier coefficients of the $F_{k_1,k_2}^{u,v}$. Having run this code, we can obtain Theorem 1.1.

Proof of Theorem 1.1. Suppose (k_1, k_2) is a pair in the table above. Chenevier-Renard [CR15] have computed the dimension $m(k_1, k_2)$ of the space of algebraic modular forms on G_2^c of weight (k_1, k_2). From their computations combined with [CT20], one sees that for the (k_1, k_2) in the table, $m(k_1, k_2) = m(\lambda(k_1, k_2))$. Consider now the map from level one algebraic modular forms on G_2^c of weight $k_1\omega_1 + k_2\omega_2$ to polynomials in $v_1, v_2, v_3, w_1, w_2, w_3$ that is the composition of the theta lift with the Tth Fourier coefficient, for $T = \frac{1}{2}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Computing the Tth Fourier coefficient of a few $F_{k_1,k_2}^{u,v}$, one finds a space of polynomials of dimension at least $m(k_1, k_2) = m(\lambda(k_1, k_2))$. Consequently, the theta lift is bijective in these cases. Applying Theorem 3.1 gives the result. $\square$

Remark 3.3. The theta lift from G_2^c to Sp_6 is not expected to be surjective onto the cusp forms of weight $\lambda(k_1, k_2)$, nor is it expected to be injective in general. For example, as explained to the author by Gaetan Chenevier, for every level one cuspidal eigenform f of weight $2k$ for PGL_2, there should be an associated level one algebraic modular form on G_2^c of weight $(k-2)\omega_2$. And, moreover, when k is odd so that the central L-value $L(f, 1/2) = 0$, the Arthur multiplicity conjecture predicts that this eigenform should not lift to Sp_6, so that its theta lift is 0. As a specific example, when $k = 9$, the dimension $m(0, 7) = 1$, but computing a few Fourier coefficients of the $F_{0,7}^{u,v}(Z)$ in SAGE for various specific u, v gives 0.

3.2. **Sage implementation to find Fourier coefficients.** If $T = [c_1, c_2, c_3; x_1, x_2, x_3]$ define the projection of T to be $T_0 = [c_1, c_2, c_3, \frac{1}{2}\operatorname{tr}_{\mathbb{O}}(x_1), \frac{1}{2}\operatorname{tr}_{\mathbb{O}}(x_2), \frac{1}{2}\operatorname{tr}_{\mathbb{O}}(x_3)]$, which is a half-integral symmetric matrix. To calculate the T_0 Fourier coefficient of $F_{k_1,k_2}^{u,v}(Z)$ on Sp_6, one must sum $\sigma_3(d_T)P_{k_1,k_2}(T; u, v)$ for all rank one $T \in J_R$ whose projection is T_0. The SAGE code will do this for those T_0 with $c_1 = 1$. Note that in this case, d_T is always equal to 1, so the factor $\sigma_3(d_T) = 1$.

To find those $T \in J_R^{rk=1}$ with $\operatorname{proj}(T) = T_0$, the code implicitly uses the following easy lemma.

Lemma 3.4. *Suppose $T = [1, b, c; x_1, x_2, x_3] \in J$. Then T is rank one if and only if $n(x_2) = c$, $n(x_3) = b$ and $x_1 = (x_2 x_3)^*$.*

The SAGE code takes in an element $T_0 \in S_3(\mathbf{Z})^{\vee}$, with $c_1(T_0) = 1$ and finds all rank one T with projection equal to T_0 using Lemma 3.4. Then, for each such T, the code computes $P_{k_1,k_2}(T; u, v)$ for a given choice of u, v, provided by the user, and sums up the results. The output is a polynomial in the variables $v_1, v_2, v_3, w_1, w_2, w_3$. To aid the user in constructing null pairs u, v, the user need only enter 6 elements of K, $[r_1, r_2, r_3, r_4, r_5, r_6]$. The code will then produce for you $u_{new} = \exp(n)e_3^*, v_{new} = \exp(n)e_1$, where $n = \sum_{j=1}^{6} r_j \ell_j$ in the notation of Lemma 2.1. The elements u_{new}, v_{new} again form a null pair.

4. Quaternionic modular forms on G_2

In this section, we give our results on quaternionic modular forms on split G_2. We begin by explaining some of the results of [Pol23] in this setting, and then explain the proof of Theorem 1.2. The proof of this theorem uses SAGE computations, which are also explained in this section.

4.1. **Fourier expansion of the quaternionic exceptional theta lift.** To set up the first result, recall $K = \mathbf{Q}(\sqrt{-1})$. Let $J^{\vee}$ denote the $\mathbf{Q}$ dual of J, and let $J_R^{\vee}$ denote the $\mathbf{Z}$-dual of J_R. Set $W_J = \mathbf{Q} \oplus J \oplus J^{\vee} \oplus \mathbf{Q}$, and define $W_J(\mathbf{Z}) = \mathbf{Z} \oplus J_R \oplus J_R^{\vee} \oplus \mathbf{Z}$, so that $W_J(\mathbf{Z})$ is a lattice in the 56-dimensional $\mathbf{Q}$ vector space W_J. If $w = (a, b, c, d) \in W_J$, define $pr_I(w)(u, v)$ to be the binary cubic form given as $pr_I(w)(u, v) = au^3 + (b, I^{\#})u^2 v + (c, I)uv^2 + dv^3$. Set $J_K = J \otimes K$ and J_K^0 to be the trace 0 elements of J_K. For $w \in W_J(\mathbf{Z})$, let d_w be the largest integer so that $d_w^{-1}w \in W_J(\mathbf{Z})$. If $X, Y \in J_K$, set $P_{m,I}(w; X, Y) = ((b, X)_I(c, Y) - (b, Y)_I(c, X))^m$. Here $(c, Y), (c, X)$ indicate the natural pairing between J and $J^{\vee}$. Finally, recall that V_1 denotes the irreducible representation of F_4^c of dimension 273.

Theorem 4.1 (See [Pol23]). *Suppose $X, Y \in J_K^0$ are singular in the sense that $X \wedge Y \in \wedge^2 J_K^0$ is a highest weight vector of V_1 for some Borel. Let $m \geq 1$ be an integer. Then there is a cuspidal quaternionic modular form $\Theta_I(X, Y; m)$ on G_2 of level one and of weight $4 + m$ with Fourier expansion*

$$\Theta_I(X, Y; m)_Z(g) = \sum_{w=(a,b,c,d) \in W_J(\mathbf{Z})^{rk=1}} \sigma_4(d_w) P_{m,I}(w; X, Y) W_{pr_I(w), 4+m}(g).$$

For the cubic norm structure (J_R, E), there is an analogous result. Fix $\delta \in M_J^1(\mathbf{Q})$ with $\delta E = I$, as in subsection 2.2. Set $(u, v)_E = \frac{1}{4}(E, E, u)(E, E, v) - (E, u, v)$. For $w = (a, b, c, d) \in W_J$, define $pr_E(w) = au^3 + (b, E^{\#})u^2 v + (c, E)uv^2 + dv^3$. Similar to the above, define $P_{m,E}(w; X_E, Y_E) = ((b, X_E)_E(c, Y_E) - (b, Y_E)_E(c, X_E))^m$.

Theorem 4.2 (See [Pol23]). *Suppose $X, Y \in J_K^0$ are singular in the sense that $X \wedge Y \in \wedge^2 J_K^0$ is a highest weight vector of V_1 for some Borel. Let $m \geq 1$ be an integer, and set $X_E = \delta^{-1}X$, $Y_E = \delta^{-1}Y$. Then there is a cuspidal quaternionic modular form $\Theta_E(X_E, Y_E; m)$ on G_2 of level one and of weight $4 + m$ with Fourier expansion*

$$\Theta_E(X_E, Y_E; m)_Z(g) = \sum_{w=(a,b,c,d) \in W_J(\mathbf{Z})^{rk=1}} \sigma_4(d_w) P_{m,E}(w; X_E, Y_E) W_{pr_E(w), 4+m}(g).$$

Remark 4.3. The forms $\Theta_I(X, Y; m)$ and $\Theta_E(X, Y; m)$ arise as theta lifts from F_4^c. If $m \geq 1$, it is reasonable to expect that they span the set of level one cuspidal quaternionic modular forms of weight $4 + m$ on G_2. However, we do not know this at the time of writing. What one knows is that the cuspidal modular forms $\Theta_I(X, Y; m)$ and $\Theta_E(X, Y; m)$ can be generalized to arbitrary level, and that the general-level versions of these functions span the space of all cuspidal quaternionic modular forms of even weight at least 6. However, we do not know, at this time, the analogous statement in odd weight. And, in even weight, it could be that a linear combination of these "general level" analogues of $\Theta_I(X, Y; m)$ and $\Theta_E(X, Y; m)$ is needed to produce some fixed level one form. That being said, the data we have collected at this time supports the hypothesis that the $\Theta_I(X, Y; m)$ and $\Theta_E(X, Y; m)$ span the level one, weight at least 5 cuspidal modular forms on G_2.

Fix an integral binary cubic form $f(u, v) = au^3 + bu^2v + cuv^2 + dv^3$. Let $\Omega_{f,I} = \{w \in W_J(\mathbf{Z}) : \mathrm{rk}(w) = 1, pr_I(w) = f\}$ and likewise for $\Omega_{f,E}$. Observe that the f-Fourier coefficient of $\Theta_I(X, Y; m)$ is a sum over elements of the set $\Omega_{f,I}$, and likewise the f-Fourier coefficient of $\Theta_E(X_E, Y_E; m)$ is a sum over elements of the set $\Omega_{f,E}$. In order for the above theorems to make sense, one needs to know that the sets $\Omega_{f,I}$ and $\Omega_{f,E}$ are finite. This is true. To make the SAGE implementation work, we need an explicit form of this finiteness, which we give in the following lemma for the case when the binary cubic form f is monic, i.e., when $a = 1$.

Lemma 4.4. *Suppose $f(u, v)$ is as above, with $a = 1$. Then*

$$\Omega_{f,I} = \{(1, T, T^\#, n_J(T)) : (T, I^\#) = b, (T^\#, I) = c, n_J(T) = d\}$$

and

$$\Omega_{f,E} = \{(1, T, T^\#, n_J(T)) : (T, E^\#) = b, (T^\#, E) = c, n_J(T) = d\}.$$

Moreover, if $(1, T, T^\#, n_J(T)) \in \Omega_{f,I}$, then $(T, T)_I = b^2 - 2c$. Likewise, if $(1, T, T^\#, n_J(T)) \in \Omega_{f,E}$, then $(T, T)_E = b^2 - 2c$.

Note that the lemma implies the finiteness in an explicit way, because the quadratic forms $(T, T)_I$ and $(T, T)_E$ are positive-definite. The lemma is an easy consequence of properties of rank one elements (see, for instance, [Pol21a]) and the definitions.

We can now explain the proof of Theorem 1.2, which uses our SAGE implementation to compute finitely many Fourier coefficients of the $\Theta_E(X_E, Y_E; m)$.

Proof of Theorem 1.2. Let $m = 5$ or $m = 7$. Computing a single Fourier coefficient of $\Theta_E(X_E, Y_E; m)$ for a somewhat randomly chosen X_E, Y_E, one sees that these quaternionic modular forms are nonzero. Thus, by Dalal's dimension formula [Dal23], these quaternionic modular forms must span $S_9(G_2(\mathbf{Z}))$ and $S_{11}(G_2(\mathbf{Z}))$. We prove in general in [Pol23] that the span of the $\Theta_E(X_E, Y_E; m)$ and $\Theta_I(X, Y; m)$ have integral Fourier coefficients, so this gives the first part of the theorem.

For the part of the theorem regarding the Fourier coefficients corresponding to cubic rings $\mathbf{Z} \times B$, it is always true that all Fourier coefficients of $\Theta_E(X_E, Y_E; m)$ corresponding to cubic rings $S = \mathbf{Z} \times B$ are 0. To see this, we have that if $f(u, v)$ is an integral binary cubic form, then the f Fourier coefficient of $\Theta_E(X_E, Y_E; m)$ is a sum over the set $\Omega_{f,E}$. However, when f corresponds to a cubic ring of the form $\mathbf{Z} \times B$, the set $\Omega_{f,E}$ turns out to be empty. This latter fact follows from [EG96, Proposition 5.5]. (This is special to $\Omega_{f,E}$, it does not apply to $\Omega_{f,I}$.). This concludes the proof. $\square$

Besides functions that compute finitely many Fourier coefficients of the $\Theta_I(X, Y; m)$ and the $\Theta_E(X_E, Y_E; m)$, the SAGE file `g2_motives.sage` also includes the function `Dalal_dim_k(k)`, that takes Dalal's explicit formula [Dal23] for the dimension of the space of level one QMFs on G_2 of weight $k \geq 3$ and puts it into the computer. The smallest weight in which there is a nonzero cusp form, according to Dalal's formula, is in weight $k = 6$. In this case, the space of weight 6 level one quaternionic modular forms is one-dimensional, spanned by an element F_6. Computing with the

SAGE code, one finds the F_6 can be normalized to have the following Fourier coefficients. (The computation took about one hour on my laptop.). In the table, the ordered 3-tuple is the (b, c, d) of the monic binary cubic $f(u, v) = u^3 + bu^2v + cu^2v + dv^3$.

$f(u,v) = u^3 + bu^2v + cu^2v + dv^3$	$a(f)$
$(0, -3, -1)$	48600
$(0, -3, 0)$	1620
$(0, -2, -1)$	15
$(0, -2, 0)$	1680
$(0, -1, 0)$	-7
$(1, -3, -3)$	-10080
$(1, -3, -2)$	25575
$(1, -3, -1)$	28800
$(1, -3, 0)$	-1485
$(1, -2, -2)$	-30
$(1, -2, -1)$	12600
$(1, -2, 0)$	-63

4.2. **SAGE implementation:** (J_R, I). To make the formulas used in Theorem 4.1 explicit, so that it can be put into SAGE, one uses the following easy lemma.

Lemma 4.5. *Let* $T = [u_1, u_2, u_3, v_1, v_2, v_3]$, $X = [x_0, x_1, x_2; x_3, x_4; x_5]$ *and* $Y = [y_0, y_1, y_2; y_3, y_4, y_5]$.
  (1) $(T, T)_I = u_1^2 + u_2^2 + u_3^2 + 2n(v_1) + 2n(v_2) + 2n(v_3)$.
  (2) $(T, I^{\#}) = u_1 + u_2 + u_3$
  (3) $(T^{\#}, I) = u_1u_2 + u_2u_3 + u_3u_1 - n(v_1) - n(v_2) - n(v_3)$
  (4) $\det(T) = u_1u_2u_3 - u_1n(v_1) - u_2n(v_2) - u_3n(v_3) + (v_1, v_2, v_3)_{\mathbb{O}}$.
  (5) $(X, T)_I = x_0u_1 + x_1u_2 + x_2u_3 + (v_1, x_3) + (v_2, x_4) + (v_3, x_5)$
  (6) $(X, T^{\#}) = A_1 - A_2 + A_3$ *where*
      (a) $A_1 = x_0(u_2u_3 - n(v_1)) + x_1(u_3u_1 - n(v_2)) + x_2(u_1u_2 - n(v_3))$
      (b) $A_2 = u_3(x_5, v_3) + u_2(x_4, v_2) + u_1(x_3, v_1)$
      (c) $A_3 = (x_3, v_2, v_3) + (x_4, v_3, v_1) + (x_5, v_1, v_2)$.
One has similar formulas for $(Y, T)_I$ *and* $(Y, T^{\#})$.

Now every piece of the computation of $P_{m,I}((1, T, T^{\#}, n_J(T)); X, Y)$ is completely explicit, including how to find all T with $pr_I((1, T, T^{\#}, n_J(T))) = f(u, v)$. Indeed, to find all such T, it suffices to find all $u_1, u_2, u_3 \in \mathbf{Z}$ and $v_1, v_2, v_3 \in R$ with $u_1^2 + u_2^2 + u_3^2 + 2n(v_1) + 2n(v_2) + 2n(v_3) = b^2 - 2c$. Because this quadratic form is visibly decomposable, to find such u_i and v_j, one can do it piecewise for the quadratic forms $x \mapsto x^2$ for $x \in \mathbf{Z}$ and $v \mapsto n_{\mathbb{O}}(v)$ for $v \in R$. This fact leads to our implementation to compute the Fourier coefficients of Θ_I being faster than its Θ_E counterpart.

To find various suitable singular pairs X, Y to use as inputs, the code has implemented the exponential of the elements $\Phi'_{u \wedge v}$ of Lemma 2.5.

4.3. **SAGE implementation:** (J_R, E). To implement the formulas in Theorem 4.2, one uses the following straightforward lemma:

Lemma 4.6. *Suppose* $T = [c_1, c_2, c_3; x_1, v_2, v_3]$. *Then* $(E^{\#}, T) = 2(c_1 + c_2 + c_3) + (\beta^*, x_1 + v_2 + v_3)$ *and*

$$(E, T^{\#}) = 2(c_1c_2 + c_2c_3 + c_3c_1 - n(x_1) - n(v_2) - n(v_3)) - (\beta, c_1x_1 + c_2v_2 + c_3v_3)$$
$$+ (\beta, v_2, v_3) + (\beta, v_3, x_1) + (\beta, x_1, v_2).$$

Moreover, the value $(T, X)_E$ *can be computed using the Gram matrix for the quadratic form* $(T, T)_E$.

Remark 4.7. The x_1 in the above lemma is not a typo; we have called that octonion x_1, instead of v_1, so that the variables used in the lemma are similar to those used in the SAGE code (which, unfortunately, are not completely parallel.)

To compute the $f(u,v)$ Fourier coefficient of $\Theta_E(X_E, Y_E; m)$, one first finds all $T \in J_R$ with $(T,T)_E = b^2 - 2c$. This uses SAGE's `short_vector_list_up_to_length` function. Then, having found all such $T = [c_1, c_2, c_3; x_1, x_2, x_3]$, we group them by those that have the same c_1, c_2, c_3 and x_1. This allows some of the computation implicit in Lemma 4.6 that is identical for multiple T to be done once, instead of repeatedly. SAGE computes each $P_{m,E}(w; X_E, Y_E)$ and sums up the results.

To find various suitable singular pairs X_E, Y_E one again uses the exponential of the elements $\Phi'_{u \wedge v}$ of Lemma 2.5, together with the element δ_E^{-1}, which is also implemented in SAGE.

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