ALGORITHMS FOR p-ADIC HEIGHTS ON HYPERELLIPTIC CURVES OF ARBITRARY REDUCTION

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ABSTRACT. In this paper, we develop an algorithm for computing Coleman–Gross (and hence Nekovář) p-adic heights on hyperelliptic curves over number fields with arbitrary reduction type above p. This height is defined as a sum of local heights at each finite place and we use algorithms for Vologodsky integrals, developed by Katz and the second-named author, to compute the local heights above p. We also discuss an alternative method to compute these for odd degree genus 2 curves via p-adic sigma functions, via work of the first-named author. For both approaches one needs to choose a splitting of the Hodge filtration. A canonical choice for this is due to Blakestad in the case of an odd degree curve of genus 2 that has semistable ordinary reduction at p. We provide an algorithm to compute Blakestad's splitting, which is conjecturally the unit root splitting for the action of Frobenius. We give several numerical examples, including the first worked quadratic Chabauty example in the literature for a curve with bad reduction.

CONTENTS

1. INTRODUCTION

Fix once and for all a prime number p. In the literature, there are several definitions of p-adic height pairings on abelian varieties over number fields, for instance due to Mazur–Tate [MT83]. These are bilinear maps $A(F) \times A(F) \to \mathbb{Q}_p$, where A is an abelian variety over a number field F. Most of these constructions are quite similar to constructions of the real-valued Néron–Tate height pairing. Algorithms for computing p-adic heights

- allow one to compute p-adic regulators, some of which fit into p-adic versions of the Birch and Swinnerton–Dyer conjecture ([MTT86, MST06, SW13, BMS16]), and
- play a crucial role in carrying out the quadratic Chabauty method for integral (see [BBM16]) and rational (see [BD18]) points on curves.

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The p-adic height pairing h constructed by Coleman–Gross [CG89], Colmez [Col98] and Besser [Bes22] for the case of a Jacobian variety J of a curve C/F has the advantage that it can be described solely in terms of C. In order to define this pairing (see §2), we first need to choose, as for most constructions of p-adic heights,

- a continuous idèle class character $\chi = (\chi_v)_v \colon \mathbb{A}_F^\times/F^\times \to \mathbb{Q}_p$, and
- for each v such that χ_v is ramified, a subspace $W_v \subset H^1_{\mathrm{dR}}(C_v/F_v)$ that is complementary to the subspace of holomorphic forms.

Here F_v is the completion of F at a finite place v and C_v is the base change of C to F_v. When the Jacobian variety J_v of C_v has semistable ordinary reduction, there is a natural choice for W_v, namely the unit root subspace for the action of the Frobenius endomorphism.

The pairing h is, by definition, a sum of local height pairings h_v between divisors on C_v of degree 0 with disjoint support, and the nature of h_v depends on χ_v:

- When χ_v is unramified, h_v is defined in terms of arithmetic intersection theory (see (2.2)) and its computation is standard; see Remark 3.3.
- When χ_v is ramified, on the other hand, the local pairing h_v is defined using Vologodsky's theory of p-adic integration [Vol03]. It can be computed as explained in [BB12, §5] and [GM23] if C is a hyperelliptic curve of good reduction at v, where the Vologodsky integrals are in fact Coleman integrals. In Section 3, we explain how to remove the good reduction assumption from this setting (see Algorithm 1) using the algorithms in [Kay22, KK22] developed by Katz and the second-named author. Since by work of Besser [Bes04, Bes22], h_v is equivalent to a special case of the local p-adic height constructed by Nekovář [Nek93] using p-adic Hodge theory, we also obtain an algorithm to compute the latter on hyperelliptic curves.

In Section 4, we discuss the case of a genus 2 curve C/F. In this case, the first-named author showed in [Bia23a] (using work of Colmez [Col98]) that h_v can be expressed in terms of v-adic sigma functions constructed in [Bia23a, Section 3], extending work of Blakestad [Bla18]. This leads to an alternative algorithm to compute h_v, implemented by the first-named author and available from [Bia23b]. In our earlier work [BKM23], we extended some of the results in [Bia23a] by expressing an extension of the Coleman–Gross height to certain divisors with *common* support in terms of v-adic sigma functions. This then leads to an algorithm for the computation of such heights.

We also showed in [BKM23] that on general abelian varieties, p-adic Néron functions give a local decomposition of a special case of the p-adic height constructed by Mazur and Tate [MT83], depending on the same data as h. When χ_v is ramified and the abelian variety has semistable ordinary reduction at v, then Mazur and Tate constructed a *canonical* local height pairing, which is of particular importance; for instance, the p-adic regulator in the p-adic Birch and Swinnerton-Dyer conjecture of Mazur–Tate–Teitelbaum for elliptic curves with good ordinary and non-split multiplicative reduction is defined in terms of it. When J is the Jacobian of a genus 2 curve C/F given by a quintic model that is semistable at v such that J has ordinary reduction at v, Blakestad [Bla18] constructed a complementary subspace W_v^C (see §4.2) such that the associated local Coleman–Gross height gives the canonical local Mazur–Tate height (see [BKM23, Theorem 5.23]). As shown by the first-named author, W_v^C is in fact the unit root subspace when the reduction is good ordinary; see [Bia23a, Proposition 3.8]. We believe that this is still the case if reduction is semistable ordinary; see [BKM23, Conjecture 3.17]. In §4.2.1, we explain how to compute W_v^C numerically.

In Section 5, we provide a number of numerical examples to illustrate our algorithms. We show that various identities of global heights predicted by identities in the Jacobian and comparison results between different constructions of local p-adic heights hold to our working precision. Moreover, we describe the first worked quadratic Chabauty example for a curve with bad reduction at p. Namely, we show in Example 5.5 that the integral points on the genus 2 curve $C\colon y^2 = x^5 + x^3 - 2x + 1$ are precisely $(0, \pm 1), (\pm 1, \pm 1)$ by running quadratic Chabauty, as described in §4.1.1, at the prime $p = 5$ of bad reduction for C. This example uses the techniques from [Bia23a]. Our examples are computed using `SageMath` [The24] and `Magma` [BCP97]. Our code can be found at

`https://github.com/KayaEnis/padicHeights`.

1.1. **Notation.** Let F denote a number field with ring of integers $\mathcal{O}_F$. For a finite place v of F, we write F_v and $\mathcal{O}_v$ for the completions at v. For each such v, fix a uniformizer π_v of $\mathcal{O}_v$ and let $\log_v\colon \mathcal{O}_v^\times \to F_v$ be the unique homomorphism extending

$$\log_v\colon 1 + \pi_v \mathcal{O}_v \to F_v, \quad 1 + z \mapsto \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}.$$

Let C be a smooth, projective and geometrically integral curve defined over F, and let J/F be its Jacobian variety. By $\mathrm{Div}^0(C)$, we mean the group of degree 0 divisors on C defined over F. For a finite place v of F, we denote $C_v := C \otimes F_v$.

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2. Coleman–Gross heights

The construction of the (extended) Coleman–Gross height pairing [CG89, Bes22] is described in detail in [BKM23, §2]. In this section, we give a brief summary for the convenience of the reader.

The pairing, which we denote by h, is a function from $\mathrm{Div}^0(C) \times \mathrm{Div}^0(C)$ to $\mathbb{Q}_p$. It depends on some choices. First, let $\chi = (\chi_v)_v\colon \mathbb{A}_F^\times / F^\times \to \mathbb{Q}_p$ be a continuous idèle class character. If χ_v is unramified, then by continuity, χ_v is determined by $\chi_v(\pi_v)$. On the other hand, if χ_v is ramified, then we have $v \mid p$ and we may decompose χ_v on $\mathcal{O}_v^\times$ as

$$\chi_v = t_v \circ \log_v\,, \tag{2.1}$$

where t_v is a $\mathbb{Q}_p$-linear map from F_v to $\mathbb{Q}_p$. Then there is a unique branch $\log_v\colon F_v^\times \to F_v$ such that (2.1) extends to $F_v^\times$. Second, for each v such that χ_v is ramified, let W_v be a subspace of $H^1_{\mathrm{dR}}(C_v/F_v)$ that is complementary to the space of classes of holomorphic forms. We remark that, when C_v has semistable ordinary reduction in the sense of [MT83, §1.1], there is a canonical choice of such a complementary space: the unit root subspace for the action of the Frobenius endomorphism.

We now describe the pairing $h = h_{\chi,\{W_v\}_v}$. For two elements D_1 and D_2 in $\mathrm{Div}^0(C)$, we have

$$h(D_1, D_2) = \sum_v h_v(D_1 \otimes F_v, D_2 \otimes F_v)\,,$$

where the sum is over finite places of F. Here h_v is the local height pairing at v defined below, where we assume that D_1 and D_2 have disjoint support.

Remark 2.1. One can still define $h_v(D_1, D_2)$ for D_1 and D_2 with common support thanks to the work of Balakrishnan–Besser [BB15]. This involves choosing a tangent vector at each point in $\mathrm{Supp}(D_1) \cap \mathrm{Supp}(D_2)$. Although the local terms depend on the tangent vectors, the global height pairing does not, provided that we make these choices consistently at all places. See [BB15, Sections 2 and 3] or [BKM23, §2.2] for details.

When χ_v is unramified, the local term is described using arithmetic intersection theory; more precisely, by the proof of [CG89, Proposition 1.2], we have

$$h_v(D_1, D_2) = \chi_v(\pi_v) \cdot (\mathcal{D}_1 \cdot \mathcal{D}_2), \tag{2.2}$$

where $(\mathcal{D}_1 \cdot \mathcal{D}_2)$ denotes the intersection multiplicity of certain extensions $\mathcal{D}_1$ and $\mathcal{D}_2$ of D_1 and D_2, respectively, to a proper regular model of C over $\mathcal{O}_v$. The pairing $h_v(D_1, D_2)$ is continuous, bi-additive and symmetric.

The local term at a place v such that χ_v is ramified (which implies that $v \mid p$) is given in terms of a Vologodsky integral ${}^{\mathrm{Vol}}\!\!\int$ (see [Vol03, Theorem B] and [Bes05, Theorem 2.1]). We need some notation first. We say a differential of the third kind, if it is regular except possibly for simple poles with integer residues. The differentials of the third kind on C_v defined over F_v form a group $T(F_v)$. The residue divisor homomorphism $\mathrm{Res}\colon T(F_v) \to \mathrm{Div}^0(C_v)$ is given by $\nu \mapsto \sum_P \mathrm{Res}_P \nu \cdot (P)$ where the sum is taken over closed points of C_v, and by [CG89, Proposition 2.5], there is a canonical homomorphism $\Psi\colon T(F_v) \to H^1_{\mathrm{dR}}(C_v/F_v)$ which is the identity on holomorphic differentials. Then there exists a unique form ω_{D_1} in $T(F_v)$ satisfying $\mathrm{Res}(\omega_{D_1}) = D_1$ and $\Psi(\omega_{D_1}) \in W_v$, and the local pairing is defined by

$$h_v(D_1, D_2) = t_v\left({}^{\mathrm{Vol}}\!\!\int_{D_2} \omega_{D_1}\right).$$

Here we use the branch $\log_v$ of the p-adic logarithm determined by the extension of (2.1) to $F_v^\times$. The pairing $h_v(D_1, D_2)$ is continuous and bi-additive, but not symmetric in general. It is symmetric when the subspace W_v is isotropic with respect to the cup product pairing $\cup$ on $H^1_{\mathrm{dR}}(C_v/F_v)$; see [CG89, Proposition 5.2].

We finish this section by noting that the global pairing h on C factors through the Jacobian variety J, since by [CG89, Proposition 1.2, Proposition 5.2] we have $h_v(D, \mathrm{div}(f)) = \chi_v(f(D))$ for any $D \in \mathrm{Div}^0(C_v)$ and $f \in F_v(C_v)^\times$ without zeros or poles along D. That is to say the pairing $h = h_{\chi,\{W_v\}_v}$ induces a bilinear pairing $h\colon J(F) \times J(F) \to \mathbb{Q}_p$. This pairing is symmetric if all W_v are isotropic with respect to the cup product.

3. Computing Coleman–Gross heights

We keep the notation introduced in §2. Additionally, for a differential form ρ of the second kind on C_v, denote by $[\rho]$ its cohomology class in $H^1_{\mathrm{dR}}(C_v/F_v)$.

3.1. **Local and global symbols.** A crucial step in computing $h_v(D_1, D_2)$, for a place v with χ_v ramified, is the construction of the form ω_{D_1}. This requires the explicit computation of the homomorphism Ψ, but its original definition is not suitable for this. As in [BB12], we express this map in terms of Besser's *local* and *global symbols* in order to make it more explicit.

Definition 3.1. Let ω be a form of the third kind on C_v. Let ρ be a form of the second kind on C_v and fix a point Z at which ρ is regular. The **local symbol** at a point $A \in C_v$ is defined as

$$\langle \omega, \rho \rangle_A := -\mathrm{Res}_A \left(\omega \int^{\mathrm{Vol}} \rho \right),$$

where by ${}^{\mathrm{Vol}}\!\int \rho$ we mean the formal integral of a local expansion of ρ around A whose constant term is the Vologodsky integral ${}^{\mathrm{Vol}}\!\int_Z^A \rho$ (resp. ${}^{\mathrm{Vol}}\!\int_Z^{A'} \rho$ for a nearby point A') if ρ is regular (resp. singular) at A[1]. Then the **global symbol** is defined as

$$\langle \omega, \rho \rangle := \sum_A \langle \omega, \rho \rangle_A,$$

where A runs over all points where either ω or ρ has a singularity.

The following result is due to Besser and allows us to compute Ψ by computing global symbols.

Proposition 3.2. ([Bes05, Corollary 3.14]) *Let ω and ρ be as in Definition 3.1. Then the global symbol $\langle \omega, \rho \rangle$ is nothing but the cup product $\Psi(\omega) \cup [\rho]$.*

3.2. **Computing Coleman–Gross heights.** In this subsection, we present an algorithm for computing Coleman–Gross p-adic height pairings on hyperelliptic curves.

From now on, we assume that $p > 2$ and that our fixed curve C is a genus-g hyperelliptic curve over F with affine model

$$y^2 = b(x), \quad b(x) \in \mathcal{O}_F[x] \text{ is monic of degree } d.$$

Choose $(\chi, \{W_v\}_v)$ as in §2.

Remark 3.3. In [BMS16, §3.1], it is already explained how to compute the local height pairing h_v at places v with χ_v unramified; see [Hol12, Mül14, VBHM20] for a detailed account.

Therefore our main interest lies in the local height pairings at places v of ramifications for χ, where p-adic integration is used. Fix a finite place v of F with the property that χ_v is ramified. Recall that this can only happen for primes v above p and that from χ_v we obtained a branch $\log_v$ of the p-adic logarithm and a linear map t_v.

[1]When $\mathrm{Res}_A(\omega) = 0$ (in particular, when ω is regular at A), the choice of constant of integration does not matter.

Algorithm 1: Computing local heights at ramified places

Input:

- A finite place $v \mid p$ of F such that χ_v is ramified.
- A complementary subspace W_v, a branch $\log_v$ of the logarithm and a linear map t_v.
- Divisors $D_1, D_2 \in \mathrm{Div}^0(C_v)$ with disjoint support.

Output: The local height pairing $h_v(D_1, D_2)$.

(1) Choose a differential ω on C_v of the third kind defined over F_v with residue divisor D_1 (see §3.2.1).
(2) Determine the holomorphic form η such that $\Psi(\omega - \eta)$ lies in the complementary subspace W_v (see §3.2.3); then $\omega_{D_1} = \omega - \eta$.
(3) Compute the Vologodsky integral ${}^{\mathrm{Vol}}\!\!\int_{D_2} \omega_{D_1}$ as described in [KK22, Kay22] with respect to $\log_v$ (see §3.2.4).
(4) Return $t_v\left({}^{\mathrm{Vol}}\!\!\int_{D_2} \omega_{D_1}\right)$.

Below, ∞^+ and ∞^- stand for $(1:1:0)$ and $(1:-1:0)$ when d is even, and ω_i denotes the 1-form $x^i \frac{dx}{2y}$.

3.2.1. *Constructing a form with given residue divisor.* Let D be a divisor of degree 0 on C_v over F_v and write $D = \sum_j ((P_j) - (Q_j))$ with points P_j, Q_j on C_v. Then $\omega = \sum_j \nu_j$ is a form of the third kind such that $\mathrm{Res}(\omega) = D$, where ν_j is defined by

$$\nu_j := \begin{cases} \left(\dfrac{y + y(P_j)}{x - x(P_j)} - \dfrac{y + y(Q_j)}{x - x(Q_j)}\right) \dfrac{dx}{2y} & \text{if } P_j \text{ and } Q_j \text{ are finite;} \\ \dfrac{y + y(P_j)}{x - x(P_j)} \dfrac{dx}{2y} & \text{if } d \text{ is odd, } P_j \text{ is finite, } Q_j = \infty; \\ 2\omega_g & \text{if } d \text{ is even, } P_j = \infty^-,\ Q_j = \infty^+; \\ \dfrac{y + y(P_j)}{x - x(P_j)} \dfrac{dx}{2y} - \omega_g & \text{if } d \text{ is even, } P_j \text{ is finite, } Q_j = \infty^-; \\ \dfrac{y + y(P_j)}{x - x(P_j)} \dfrac{dx}{2y} + \omega_g & \text{if } d \text{ is even, } P_j \text{ is finite, } Q_j = \infty^+. \end{cases}$$

Note that the points P_j, Q_j, and hence the form ν_j, are not necessarily defined over F_v, but since D is defined over F_v, so is ω.

3.2.2. *Computing* Ψ. For $j = 0, 1, \dots, 2g-1$, define

$$\rho_j := \begin{cases} \omega_j & \text{if } j = 0, \dots, g-1; \\ \omega_j & \text{if } j = g, \dots, 2g-1 \text{ and } d \text{ is odd;} \\ \omega_{j+1} + 2\mathrm{Res}_{\infty^+}(\omega_{j+1})\omega_g & \text{if } j = g, \dots, 2g-1 \text{ and } d \text{ is even.} \end{cases}$$

By construction, $\rho_0, \dots, \rho_{g-1}$ span the space of holomorphic differentials and each ρ_j is of the second kind, so that the class $[\rho_j]$ is an element of $H^1_{\mathrm{dR}}(C_v/F_v)$. Moreover, the elements $[\rho_0], \dots, [\rho_{2g-1}] \in H^1_{\mathrm{dR}}(C_v/F_v)$ are linearly independent, which implies that the set $\mathcal{B} = \{[\rho_0], \dots, [\rho_{2g-1}]\}$ forms a basis for $H^1_{\mathrm{dR}}(C_v/F_v)$.

Now let ω be a form of the third kind defined over F_v. Then $\Psi(\omega) = \sum_{i=0}^{2g-1} c_i[\rho_i]$ for some constants c_i. By Proposition 3.2,

$$\langle \omega, \rho_j \rangle = \Psi(\omega) \cup [\rho_j] = \sum_{i=0}^{2g-1} c_i([\rho_i] \cup [\rho_j]), \quad j = 0, 1, \ldots, 2g-1. \tag{3.1}$$

Let N denote the cup product matrix with respect to the basis $\mathcal{B}$. From (3.1), we get

$$\begin{pmatrix} c_0 \\ \vdots \\ c_{2g-1} \end{pmatrix} = -N^{-1} \begin{pmatrix} \langle \omega, \rho_0 \rangle \\ \vdots \\ \langle \omega, \rho_{2g-1} \rangle \end{pmatrix}.$$

Therefore, in order to compute $\Psi(\omega)$, it is enough the compute the matrix N and the global symbols $\langle \omega, \rho_j \rangle$. The former is an easy task. The latter can be done using the techniques in [KK22, Kay22]; the situation is even better if the residue divisor of ω contains only affine points:

Proposition 3.4. *If the residue divisor D of ω does not contain the point(s) at infinity, we have*

$$\langle \omega, \rho_j \rangle = \begin{cases} -{}^{\mathrm{Vol}}\!\!\int_D \rho_j - \mathrm{Res}_\infty \left(\omega \, {}^{\mathrm{Vol}}\!\!\int \rho_j \right) & \text{if } d \text{ is odd,} \\ -{}^{\mathrm{Vol}}\!\!\int_D \rho_j - \mathrm{Res}_{\infty^+} \left(\omega \, {}^{\mathrm{Vol}}\!\!\int \rho_j \right) - \mathrm{Res}_{\infty^-} \left(\omega \, {}^{\mathrm{Vol}}\!\!\int \rho_j \right) & \text{if } d \text{ is even.} \end{cases}$$

Proof. This is a straightforward generalisation of [BB12, Proposition 5.12] (or rather, its corrected version in the errata [BB19]). The key observation is that the local symbol at a point A in the support of D equals

$$-(\text{the multiplicity of } D \text{ at } A) \cdot {}^{\mathrm{Vol}}\!\!\int_Z^A \rho_j$$

since ω has a simple pole at A. Here Z is a fixed point throughout the global symbol computation. Therefore, summing over all points gives $-{}^{\mathrm{Vol}}\!\!\int_D \rho_j$. □

3.2.3. *From D to ω_D.* Let ω be a form of the third kind defined over F_v with residue divisor D. Let $\eta_0, \ldots, \eta_{g-1}$ be differentials of the second kind whose classes generate W_v. Since W_v is a complementary to the space of classes of holomorphic forms, there are constants d_i, e_i such that

$$\Psi(\omega) = d_0[\rho_0] + \cdots + d_{g-1}[\rho_{g-1}] + e_0[\eta_0] + \cdots + e_{g-1}[\eta_{g-1}]. \tag{3.2}$$

Set

$$\eta = d_0\rho_0 + \cdots + d_{g-1}\rho_{g-1} \in H^0(C_v, \Omega^1_{C_v/F_v}).$$

Then $\Psi(\omega - \eta)$ lies in W_v.

Let us explain how to determine the constants d_i. First, notice that η_j/ω_0 can be represented by a polynomial in x. Without loss of generality, we may assume that

$$\deg\left(\frac{\eta_j}{\omega_0}\right) = \begin{cases} j+g & \text{if } d \text{ is odd,} \\ j+g+1 & \text{if } d \text{ is even.} \end{cases} \tag{3.3}$$

As explained in §3.2.2, we can explicitly compute the constants c_i such that

$$\Psi(\omega) = c_0[\rho_0] + \cdots + c_{2g-1}[\rho_{2g-1}]. \tag{3.4}$$

Finally, the assumption (3.3) allows us to determine the constants d_i easily by comparing (3.2) and (3.4).

3.2.4. *Computing* ${}^{\mathrm{Vol}}\!\!\int_{D_2}\omega_{D_1}$. The methods described in [KK22, Kay22] reduce the computation of Vologodsky integrals on hyperelliptic curves to the computation of Coleman integrals on hyperelliptic curves of good reduction. The latter is feasible thanks to the algorithms developed in [BBK10, BB12, Bal15]. In particular, one uses [BB12, Algorithm 4.8] to deal with differential forms of the third kind; however, this requires the good reduction hyperelliptic curve in question to have odd degree. Recent work of Gajović [GM23] with the third-named author introduced a similar, but simpler algorithm, based on [Bal15], that also works for even degree. It is restricted to *antisymmetric* differentials but is significantly faster; we make use of this algorithm when possible.

Remark 3.5. The computation of Coleman–Gross heights between divisors with common support is somewhat more difficult. For instance, suppose that C is an odd degree hyperelliptic curve. For an affine point, fix the tangent vector dual to ω_0 and fix the tangent vector dual to ω_{g-1} for $\infty \in C(F)$. According to [BBM16, Theorem 2.2], the local height function $P \mapsto h_v(P-\infty, P-\infty)$ with respect to this choice is a double Vologodsky integral from a tangential base point at ∞ to P. The algorithms for the computation of these local heights discussed in [BBM16, §4] and [BBM17, §3.3] are crucial to carry out the quadratic Chabauty method for integral points on odd degree hyperelliptic curves introduced for good reduction in [BBM16]. No algorithms for the computation of double Vologodsky integrals have been developed yet, but it should be possible to develop such an algorithm by combining the techniques of [KK22, Kay22] with the work of Katz and Litt [KL22]. In the next section, we discuss an alternative approach to the computation of these local heights via p-adic Néron functions when $g=2$.

4. p-adic Néron functions and the canonical subspace in genus 2

In this section we briefly discuss the relation between Coleman–Gross heights and the construction of Mazur–Tate, referring to [BKM23] and [Bia23a] for details. This yields a different approach to the computation of the Coleman–Gross height when the genus is 2 and $C(F_v)$ contains a Weierstrass point for all v of ramification of χ. One advantage of this approach is that it makes it possible to compute local height pairings between certain divisors with common support which are crucial for the quadratic Chabauty method for integral points in [BBM16]; see Remark 3.5. If, in addition, the reduction of C at such v is semistable ordinary in the sense of [BKM23, Definition 5.1], then this comparison also makes it possible to isolate a canonical subspace W_v^C.

4.1. **Mazur–Tate heights, p-adic Néron functions and Coleman–Gross heights.** Mazur and Tate [MT83] construct a global p-adic height pairing on an abelian variety, also depending on the choice of a continuous idèle class character $\chi = (\chi_v)_v \colon \mathbb{A}_F^\times/F^\times \to \mathbb{Q}_p$. The global pairing is a sum of local pairings, so called χ_v-splittings (see [BKM23, §3.1]). In [BKM23, §3], we introduced a notion of p-adic Néron functions and showed that they induce χ_v-splitting. When $v \mid p$ and χ_v is ramified, the p-adic Néron function depends on the choice of a splitting of the Hodge filtration of H^1_{dR}. The global Mazur–Tate height is then the sum of the p-adic Néron functions by [BKM23, Proposition 3.19].

Suppose from now on that the abelian variety is the Jacobian J of a curve C/F. In this case the p-adic Néron function, and hence the corresponding Mazur–Tate height, depends on the choice of a complementary subspace W_v as above. More precisely, for a divisor $D \in \mathrm{Div}(J)$ and a finite place v of F, the p-adic Néron function $\lambda_{D,v} \colon J(F_v)\backslash \mathrm{Supp}(D) \to \mathbb{Q}_p$ is, up to a constant, the same as the classical real-valued Néron function if χ_v is unramified;

see [BKM23, §3.1.1]. In the ramified case, we use Besser's p-adic log functions [Bes05, BMS21] on the line bundle $\mathcal{O}(D)$ to define $\lambda_{D,v}$ in [BKM23, §3.1.2].

From now on, suppose for simplicity that C/F is an odd degree hyperelliptic curve [2]. Let $\Theta \in \mathrm{Div}(J)$ be the theta divisor corresponding to the Abel–Jacobi embedding $\iota\colon C \to J$ with respect to $\infty \in C(F)$ and, for each v, let $\lambda_v := \lambda_{2\Theta,v}$. Then we showed in [BKM23, Proposition 3.19] that the global Mazur–Tate height function (with respect to χ and W_v, for ramified χ_v) is simply the sum of λ_v for all v (with the same choices). Moreover, the local Coleman–Gross height pairing between two divisors on C with disjoint support can be expressed in terms of pullbacks of λ_v. This can be deduced from [BMS21, Remark 6.10, Corollary 6.12].

We now assume that $p \geq 5$ and that C/F has genus 2, given by an affine equation

$$C\colon y^2 = b(x) = x^5 + b_1x^4 + b_2x^3 + b_3x^2 + b_4x + b_5, \quad b_1, \dots, b_5 \in \mathcal{O}_F\,. \tag{4.1}$$

In this case, we can describe the p-adic Néron function λ_v with respect to χ and W_v explicitly by work of Blakestad [Bla18] and the first-named author [Bia23a] as we now recall, referring to [Bia23a, Sections 3 and 4] for details. We identify $H^1_{\mathrm{dR}}(C_v/F_v)$ and $H^1_{\mathrm{dR}}(J_v/F_v)$ via the isomorphism induced by ι. We assume for the rest of this section that $v \mid p$ and that χ_v is ramified. First, [Bia23a, Proposition 3.5] shows that the isotropic complementary subspaces W_v are in bijection with the set of symmetric matrices $c = (c_{ij}) \in F_v^{2\times 2}$ via

$$c \mapsto \langle \eta_1^{(c)}, \eta_2^{(c)} \rangle, \tag{4.2}$$

where

$$\eta_1^{(c)} = (-3x^3 - 2b_1x^2 - b_2x + c_{12}x + c_{11})\frac{dx}{2y}, \quad \eta_2^{(c)} = (-x^2 + c_{22}x + c_{12})\frac{dx}{2y}. \tag{4.3}$$

The v-adic σ-function $\sigma_v^{(c)}$ is defined in [Bia23a, §3.2] as the unique odd power series of the form $T_1(1 + \mathcal{O}(T_1, T_2))$ in $F_v[[T_1, T_2]]$ that solves

$$D_iD_j(\log(\sigma_v^{(c)}(T))) = -X_{ij}(T) + c_{ij}, \qquad \text{for all } 1 \leq i, j \leq 2\,,$$

where D_i is the invariant derivation dual to ω_i and and the X_{ij} are certain functions on the symmetric square of C by Grant [Gra90, §4.1, §4.2]. Via Grant's explicit formal group law $T = (T_1, T_2)$ (see [Gra90] and [Bia23a, Section 2]), $\sigma_v^{(c)}$ induces a function on a subgroup $H_v \subset J(F_v)$ of finite index (see [Bia23a, Proposition 3.4]); more precisely, H_v is a subgroup of the model-dependent kernel of reduction. By [BKM23, Corollary 4.6], the p-adic Néron function with respect to the subspace W_v is then given by

$$\lambda_v(P) = -\frac{2}{m^2} \cdot \chi_v\left(\frac{\sigma_v^{c(W_v)}(T(mP))}{\phi_m(P)}\right), \quad mP \in H_v\,, m > 0 \tag{4.4}$$

for any $P \in J(F_v) \setminus (\mathrm{Supp}(\Theta) \cup J(F_v)_{\mathrm{tors}})$; here ϕ_m is the m-th division polynomial constructed by Kanayama [Kan05] and [Uch11]. Using the symmetric p-adic Green functions due to Colmez [Col98], which by [Bia23a, Theorem 5.30] are essentially p-adic Néron functions, one obtains the following formula:

[2]The local results stated below for a place v continue to hold if we only assume that $C(F_v)$ contains a Weierstrass point.

Proposition 4.1. ([BKM23, Corollary 4.7]) *For distinct $P_1, P_2, Q_1, Q_2 \in C(F_v) \setminus \{\infty\}$, we have*

$$h_v^{\mathrm{CG}}(P_1 - P_2, Q_1 - Q_2) = -\frac{1}{2} \sum_{1 \le i,j \le 2} (-1)^{i+j} \lambda_v([Q_i - P_j]),$$

with respect to the same choices χ_v and W_v on both sides.

By [Bia23a, Corollary 5.32], this extends to a formula for Coleman–Gross height pairings between general divisors with disjoint support. The main result of [BKM23] extends this as follows. Define the Coleman–Gross local height function

$$\lambda_v^{\mathrm{CG}} \colon J(F_v) \setminus \mathrm{Supp}(\Theta) \to \mathbb{Q}_p ; \quad [Q_1 - Q_2] \mapsto h_v(Q_1 - Q_2, Q_1 - Q_2)$$

(see [BKM23, Definition 4.1]), where we extend the Coleman–Gross height pairing by choosing the tangent vector dual to ω_0 for the affine points Q_1, Q_2 (see Remarks 2.1 and 3.5).

Theorem 4.2. ([BKM23, Theorem 1.1]) *We have $\lambda_v = \lambda_v^{\mathrm{CG}}$ with respect to the same choices χ_v and W_v on both sides.*

Remark 4.3. The genus 1 analogue of Theorem 4.2 is due to Balakrishnan and Besser [BB15, §4]. Their proof is given for good ordinary reduction, but it extends readily to the general case. It is a classical result going back to Bernardi [Ber81] and Mazur–Tate [MT83] that global p-adic heights on elliptic curves can be expressed in terms of p-adic sigma functions. See for instance [Bia20] and the references therein for the relation of the latter to p-adic Néron functions.

An algorithm to compute λ_v was implemented by the first-named author and is available from [Bia23b]. By Proposition 4.1 and Theorem 4.2, this yields an alternative method for computing local Coleman–Gross heights at v. The algorithm needs a suitable matrix $c \in F_v^{2\times 2}$ as input. For instance, one can simply take $c = 0$ as in [Bia23a, §3.1]; a more canonical choice is described in §4.2 below. Furthermore, one may use the results of [Bia23a] to compute integrals of differentials of the first, second and third kind, see [Bia23a, §6.7].

Remark 4.4. Both Proposition 4.1 and Theorem 4.2 also hold true for primes v of F such that χ_v is unramified; in this case, both sides only depend on the local character χ_v, and not on the choice of a complementary subspace. See [Bia23a, Remark 4.6] and [BKM23, Theorem 4.2].

4.1.1. *Application to quadratic Chabauty.* Suppose now that C is a genus 2 curve given by a model (5.6) over $F = \mathbb{Q}$ with Jacobian $J/\mathbb{Q}$ of Mordell–Weil rank 2. Suppose also that the linear extension $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) \colon J(\mathbb{Q}_p) \otimes \mathbb{Q}_p \to \mathbb{Q}_p^{\oplus 2}$ of the abelian logarithm restricts to an injective map on $J(\mathbb{Q}) \otimes \mathbb{Q}$. We normalise the character χ to be the cyclotomic character satisfying $\chi_p(p) = \log_p(p) = 0$, and we fix some complementary isotropic subspace W_p. The quadratic Chabauty method for integral points on hyperelliptic curves, introduced in [BBM16], uses properties of local and global heights to produce a locally analytic p-adic function

$$\rho \colon C(\mathbb{Z}_p) \longrightarrow \mathbb{Q}_p \tag{4.5}$$

such that $\rho(C(\mathbb{Z}))$ takes values in an effective computable finite set Γ. It was reinterpreted in the Chabauty–Kim framework [Kim09] and extended to rational points on certain curves (for instance those that satisfy, in addition, $\mathrm{rkNS}(J) > 1$) by Balakrishnan and Dogra in [BD18]. So far, all examples computed using any instance of the quadratic Chabauty method used primes p of good reduction, although the theoretical setup in [BBM16] requires no such assumption. In order to make the method from [BBM16] explicit, one would have to compute with local heights of the form $h_v(P-\infty, P-\infty)$; see Remark 3.5. As discussed in [Bia23a, Section 7], the comparison results Proposition 4.1 and Theorem 4.2 make it possible to reformulate the quadratic Chabauty method in terms of p-adic Néron functions, as we now explain.

Since Theorem 4.2 only applies to points outside the theta divisor, we cannot work directly with heights of the form $h_v(P-\infty, P-\infty)$, so we use the simple identity

$$h_v(P-\infty, P-\infty) = \frac{1}{4}\left(h_v(P-w(P), P-w(P)) + \chi_v(4y(P))^2\right) \tag{4.6}$$

proved (for $v \mid p$) in [BBM16, Section 4]. We define $\nu_v\colon C(\mathbb{Q}_v)\setminus\{P : y(P)\in\{0,\infty\}\}\to\mathbb{Q}_p$ by

$$\nu_v(P) := \lambda_v(2\iota(P)) + 2\chi_v(2y(P)) \tag{4.7}$$

where λ_p is the p-adic Néron function with respect to χ_p and W_p. From Theorem 4.2 and [BBM16, Theorem 3.1], we deduce:

Theorem 4.5. *There are constants $a_1, a_2, a_3 \in \mathbb{Q}_p$ and a finite set $\Gamma\subset\mathbb{Q}_p$, all effectively computable, such that the locally analytic function $\rho\colon C(\mathbb{Z}_p)\setminus\{P : y(P)=0\}\longrightarrow\mathbb{Q}_p$ defined by*

$$\rho(P) := \nu_p(P) - a_1\mathcal{L}_1^2(P) - a_2\mathcal{L}_1(P)\mathcal{L}_2(P) - a_3\mathcal{L}_2^2(P) \tag{4.8}$$

takes values in Γ for $P\in C(\mathbb{Z})$.

The elements of Γ are of the form $\sum_v \gamma_v$, where v runs through the bad primes of C and

$$\gamma_v \in \Gamma_v := \{-\nu_v(P) : P\in C(\mathbb{Z}_v),\ y(P)\neq 0\}.$$

To find the constants a_i in practice, one uses the fact that the global height function $P\mapsto h(P) := h(P,P)$ is a $\mathbb{Q}_p$-valued quadratic form on $J(\mathbb{Q})\otimes\mathbb{Q}$ and that the functions $\mathcal{L}_1^2, \mathcal{L}_1\mathcal{L}_2, \mathcal{L}_2^2$ span the space of such functions. The sets Γ_v may be computed either using the techniques in [BBM17, §3.4] or via the proof of [Bia23a, Lemma 7.2]. Finally, we may solve for the set of all $z\in C(\mathbb{Z}_p)$ such that $\rho(z)\in\Gamma$. Unless ρ is constant, this will result in a finite superset of $C(\mathbb{Z})$. From this, we can then try to provably compute $C(\mathbb{Z})$ using the Mordell–Weil sieve [BS10]; see, for instance, [BBM17].

Remark 4.6. In fact, the first-named author recovered in [Bia23a, Section 7] the main results of [BBM16] in a simpler and more direct way, working entirely with p-adic Néron functions. This is then used in loc. cit. to develop an explicit quadratic Chabauty method for rational points on certain bihyperelliptic genus 4 curves.

4.2. **The canonical subspace.** We now discuss briefly how to construct and compute a canonical complementary subspace W_v^C when the curve C/F given by (5.6) has semistable ordinary reduction in the following sense. See [BKM23, §5] for details.

Definition 4.7. We say that an abelian variety A/F has *semistable ordinary reduction* at a finite prime v if the connected component of the special fibre of the Néron model of A over $\mathcal{O}_v$ is an extension of an ordinary abelian variety by a torus. If $A=J$ is the Jacobian of a curve C/F given by a model (5.6), then we say that C has *semistable ordinary reduction* at v if J has ordinary reduction at v and if the Zariski closure of (5.6) in $\mathbb{P}^{1,3,1}_{\mathcal{O}_v}$ is semistable.

When the Jacobian J of C has semistable ordinary reduction at v, Mazur–Tate construct a canonical χ_v-splitting using formal completions (see [MT83, §1.9]).

We now summarise the construction of an isotropic complementary subspace W_v^C that induces the canonical Mazur–Tate splitting, referring to [BKM23, §5.2] for details. The construction is a straightforward generalisation of results of Blakestad [Bla18, Chapter 3] who assumed that C has good reduction and J has ordinary reduction.

Consider the local parameter $t = -\frac{x^2}{y}$ at $\infty \in C(F)$. For every positive integer n, there exist unique functions ϕ_n and ψ_n on C, regular away from ∞, with t-expansions

$$\phi_n = \frac{1}{t^{3p^n}} + \frac{A_n}{t^3} + \frac{B_n}{t} + I_n t + R_n t^3 + O(t^5), \quad A_n, B_n, I_n, R_n \in \mathcal{O}_F;$$
$$\psi_n = \frac{1}{t^{p^n}} + \frac{C_n}{t^3} + \frac{D_n}{t} + J_n t + S_n t^3 + O(t^5), \quad C_n, D_n, J_n, S_n \in \mathcal{O}_F;$$

see [Bla18, §3.2.1]. By [BKM23, Lemma 5.17, Proposition 5.19], the matrices

$$M_n := \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$$

are invertible over $\mathcal{O}_v$ for all n precisely when C has semistable ordinary reduction at v. If this is satisfied, then by [BKM23, Lemma 5.17], the limit

$$\begin{pmatrix} \alpha & \delta \\ \beta & \gamma \end{pmatrix} := \lim_{n\to\infty} \begin{pmatrix} \alpha_n & \delta_n \\ \beta_n & \gamma_n \end{pmatrix}; \qquad \begin{pmatrix} \alpha_n & \delta_n \\ \beta_n & \gamma_n \end{pmatrix} := M_n^{-1} \begin{pmatrix} I_n & R_n \\ J_n & S_n \end{pmatrix}$$

exists and lies in $\mathcal{O}_v^{2\times 2}$.

Proposition 4.8. ([BKM23, Theorem 5.23]) *Suppose that C has semistable ordinary reduction at v. Then the subspace W_v^C corresponding under (4.2) to*

$$\begin{aligned} c_{11}^C &:= 2b_1b_2 - b_1\alpha + b_1^2\beta + 3\delta - 3b_1\gamma + 3b_3, & c_{12}^C &:= b_2 - b_1\beta + 3\gamma, \\ c_{21}^C &:= b_2 + \alpha - b_1\beta, & c_{22}^C &:= \beta \end{aligned} \tag{4.9}$$

induces the canonical Mazur–Tate splitting.

Hence the p-adic Néron function with respect to the matrix c^C gives the local contribution at v of the canonical Mazur–Tate height function.

4.2.1. *Computing the canonical subspace.* In order to compute the canonical p-adic Néron function or the corresponding Coleman–Gross height, we need to compute the subspace W_v^C. When C_v has good reduction and J_v has ordinary reduction, it was shown by the first-named author that W_v^C is in fact the unit root subspace for the action of the Frobenius (see [Bia23a, Proposition 3.8]) so that one can use [BB12, Proposition 6.1] to compute a basis for W_v^C in a simple and efficient way. In [BKM23], we conjectured that W_v^C is still the unit root subspace if the reduction is semistable ordinary; see Conjecture 3.17 of loc. cit. If this is true, then it should be possible to compute it using the description of the action of Frobenius on H^1_{dR} due to Coleman–Iovita [CI99].

In any case, we can approximate W_v^C to n digits of precision by computing the functions ϕ_n and ψ_n up to $O(t^5)$. Blakestad's [Bla18, Proposition 22] (slightly adapted to arbitrary reduction in [BKM23, Lemma 5.17]) states that for every $k \neq 1, 3$, there is a unique function $\rho_k \in F(C)$ such that ρ_k has no poles outside ∞ and such that the expansion of ρ_k around ∞ is of the form

$$\rho_k(t) = t^{-k} + M_k t^{-3} + N_k t^{-1} + \mathcal{O}(t)\,. \tag{4.10}$$

In other words, we need to compute ρ_{3p^n} and ρ_{p^n} to precision $\mathcal{O}(t^5)$. To accomplish this, we simply follow the proof of [Bla18, Proposition 22], which is based on Riemann–Roch. For each $m \neq 1, 3$, write $m = 2i + 5j$ and set $\tilde{\rho}_m = x^i(-y)^j$. Then $\tilde{\rho}_m \in \mathcal{L}(m\infty) \setminus \mathcal{L}((m-1)\infty)$ and $\tilde{\rho}_m(t) = t^{-m} + \mathcal{O}(t^{-m+1})$, so that ρ_k is the unique linear combination of $\tilde{\rho}_2, \ldots, \tilde{\rho}_k$ with expansion (4.10) around ∞. We may compute $\rho_k(t)$ as follows:

(1) Set $\rho_k(t) := \tilde{\rho}_k(t)$.
(2) For all m from $k-1$ down to 2 set

$$\rho_k(t) := \rho_k(t) - a_m \tilde{\rho}_m(t),$$

where a_m is the coefficient of t^{-m} in $\rho_k(t)$.

Since we need a linear number of steps in k, and we need to apply this method to $k = 3p^n$ and $k = p^n$, the algorithm gets fairly slow even for moderate values of p and n. In practice, we only need to compute $x^i(t)$ for $i = 0, \ldots, 4$, and we need $(-y(t))^j$ for $j = 1, \ldots, M$, where $M = \lfloor k/5 \rfloor$ and j is odd, since for odd m, there are no even order terms in the expansions of $\tilde{\rho}_m$. Rather than computing these anew for every m, we first compute $(-y(t))^M$ and subsequently multiply by $y(t)^{-2}$. While computing ρ_{3p^n}, we also compute ρ_{p^n} along the way. To keep the computations feasible, we truncate all Laurent series whenever possible so that the final result is correct up to $O(t^5)$, and we work with integers modulo p^n.

5. Numerical examples

In this final section, we illustrate our algorithms with numerical examples. Below, the base field is $\mathbb{Q}$. Moreover, we take χ to be the cyclotomic character as in §4.1.1. Let w denote the hyperelliptic involution.

Example 5.1. (Genus 1) Let $C/\mathbb{Q}$ be an elliptic curve with split multiplicative reduction at a prime number p. The canonical height pairing of Mazur–Tate is the same as the Coleman–Gross height pairing with respect to the unit root subspace; see, for instance, [BKM23, Remark 3.20]. Here we will confirm this equality numerically in an example.

The canonical Mazur–Tate height of a point $P \in C(\mathbb{Q})$ is given in terms of the canonical p-adic sigma function σ_p of [MT91] by $2\log_p\left(\frac{e(P)}{\sigma_p(t(P))}\right)$, with notation as in [SW13, §4.2][3], and can be easily computed using `SageMath`. As a concrete example, consider the elliptic curve $C/\mathbb{Q}$ given by

$$y^2 = x^3 - 1351755x + 555015942 \tag{5.1}$$

with $P := \left(\frac{330483}{361}, \frac{63148032}{6859}\right) \in C(\mathbb{Q})$. This curve has split multiplicative reduction at $p = 43$. Using `SageMath`, we compute that the canonical Mazur–Tate height of P is

$$19 \cdot 43 + 7 \cdot 43^2 + 8 \cdot 43^3 + 2 \cdot 43^4 + 28 \cdot 43^5 + O(43^6). \tag{5.2}$$

For the two points $Q = (2523, 114912)$ and $R = (219, 16416)$ on C, let $D_Q = (Q) - (-Q)$ and $D_R = (R) - (-R)$; we then have $P = [D_Q] = [D_R]$. We now compute the Coleman–Gross height $h(D_Q, D_R)$ with respect to the unit root subspace and compare the result to (5.2).

[3]There, $\hat{h}_p$ is the Schneider height [Sch82], which is not of interest for this example.

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Away from 43, using the Magma implementation of the algorithm developed in [VBHM20], we have $\sum_{q\neq 43} h_q(D_Q, D_R) = 9 \cdot \log_{43}(2)$. For the component at $p = 43$, we first note that the unit root subspace is generated by $\alpha[\omega_0] + [\omega_1]$ where

$$\alpha = 17 + 37 \cdot 43 + 20 \cdot 43^2 + 11 \cdot 43^3 + 38 \cdot 43^4 + 6 \cdot 43^5 + O(43^6)\,;$$

this was computed in SageMath using [SW13, §4.2]. We then have

$$h_p(D_Q, D_R) = {}^{\mathrm{Vol}}\!\!\int_{-R}^{R} \omega_{D_Q} = {}^{\mathrm{Vol}}\!\!\int_{-R}^{R} \frac{y(Q)}{x - x(Q)} \frac{dx}{y} + (c_1\alpha - c_0)\, {}^{\mathrm{Vol}}\!\!\int_{-R}^{R} \omega_0$$

where $c_0 = {}^{\mathrm{Vol}}\!\!\int_{Q}^{-Q} \omega_1$ and $c_1 = {}^{\mathrm{Vol}}\!\!\int_{-Q}^{Q} \omega_0$. Using the techniques described in [KK22, Kay22], we compute

$$\begin{aligned}
{}^{\mathrm{Vol}}\!\!\int_{-R}^{R} \frac{y(Q)}{x - x(Q)} \frac{dx}{y} &= 29 \cdot 43 + 29 \cdot 43^2 + 18 \cdot 43^3 + 29 \cdot 43^4 + 3 \cdot 43^5 + O(43^6),\\
{}^{\mathrm{Vol}}\!\!\int_{-R}^{R} \omega_0 &= 12 \cdot 43^2 + 43^3 + 18 \cdot 43^4 + 40 \cdot 43^5 + O(43^6) = {}^{\mathrm{Vol}}\!\!\int_{-Q}^{Q} \omega_0\\
{}^{\mathrm{Vol}}\!\!\int_{Q}^{-Q} \omega_1 &= 18 + 31 \cdot 43 + 8 \cdot 43^2 + 16 \cdot 43^3 + 17 \cdot 43^4 + 8 \cdot 43^5 + O(43^6),
\end{aligned}$$

which give

$$h_p(D_Q, D_R) = 29 \cdot 43 + 28 \cdot 43^2 + 10 \cdot 43^3 + 39 \cdot 43^4 + 7 \cdot 43^5 + O(43^6).$$

Putting all of this together, we get

$$h(D_Q, D_R) = \sum_q h_q(D_Q, D_R) = 19 \cdot 43 + 7 \cdot 43^2 + 8 \cdot 43^3 + 2 \cdot 43^4 + 28 \cdot 43^5 + O(43^6)$$

which is the same result as in (5.2).

Remark 5.2. The curve in this example may appear very special since the coefficients of its defining polynomial are quite large, but it is not. In order to compare the canonical Mazur–Tate height with the Coleman–Gross height with respect to the unit root subspace, we needed an elliptic curve $C/\mathbb{Q}$ with split multiplicative reduction at a prime $p > 2$ that satisfies an additional constraint: existence of two distinct points $Q, R \in C(\mathbb{Q})$ such that

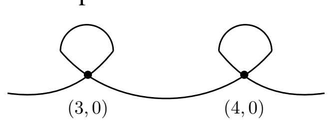
$$D_Q = (Q) - (-Q) = (R) - (-R) = D_R.$$

In this way, we could apply Algorithm 1 in order to compute the local height $h_p(D_Q, D_R)$. A quick search on the LMFDB database revealed [LMF24, 6622.i3], which is the curve in (5.1). Because the mentioned additional constraint is fairly mild, similar computations can be performed for curves given by polynomials with reasonably small coefficients.

Example 5.3. (Genus 2) Consider the hyperelliptic curve $C/\mathbb{Q}$ given by

$$y^2 = x^5 + 5x^4 - 168x^3 + 1584x^2 - 10368x + 20736.$$

Here $p = 5$ is a prime of bad semistable reduction for C, and the stable reduction is a curve of genus 0 with two ordinary double points:



Consider the points $P = (-12, 720)$, $Q = (-8, 528)$, $R = (0, -144)$, $S = (12, 432)$, $T = (36, 7920)$ on C and set

$$D_1 = (Q) - (w(Q)),\ \ D_2 = (R) - (P),\ \ D_3 = (S) - (w(P)),\ \ D_4 = (T) - (w(T)).$$

One can easily check that we have $[D_1] = 4[D_2]$ and $[D_4] = 6[D_3]$, which implies

$$6h(D_1, D_3) = h(D_1, D_4) = h(D_4, D_1) = 4h(D_4, D_2)\,, \tag{5.3}$$

provided that the chosen complementary subspace for the prime 5 is isotropic with respect to the cup product pairing. In this example, we verify this up to a certain precision.

Away from 5, using again the algorithm developed in [VBHM20]; we have

$$\sum_{q\neq 5} h_q(D_1, D_3) = 0, \qquad \sum_{q\neq 5} h_q(D_4, D_2) = -2\cdot \log_p(2) + \log_p(3).$$

For the local contributions at $p = 5$, let W_p^C be the canonical subspace. Using the algorithm described in §4.2.1, we find that this space is generated by the differentials

$$[\eta_0] = k_0[\omega_0] + k_1[\omega_1] + k_2[\omega_2] \quad \text{and} \quad [\eta_1] = l_0[\omega_0] + l_1[\omega_1] + l_2[\omega_2] + l_3[\omega_3],$$

where

$$\begin{aligned}
k_0 &= 2 + 3\cdot 5 + 4\cdot 5^2 + 2\cdot 5^3 + 5^4 + 3\cdot 5^6 + O(5^7), \\
k_1 &= 3 + 4\cdot 5 + 5^3 + 3\cdot 5^4 + 2\cdot 5^5 + 4\cdot 5^6 + O(5^7), & k_2 &= 1, \\
l_0 &= 2 + 3\cdot 5 + 2\cdot 5^2 + 4\cdot 5^3 + 5^4 + 2\cdot 5^5 + 5^6 + O(5^7), & l_2 &= 15, \\
l_1 &= 4 + 2\cdot 5 + 2\cdot 5^2 + 2\cdot 5^3 + 2\cdot 5^4 + 3\cdot 5^5 + O(5^7), & l_3 &= 3.
\end{aligned}$$

Let us compute $h_p(D_1, D_3)$. The form $\omega := \frac{y(Q)}{x - x(Q)}\frac{dx}{y}$ satisfies $\mathrm{Res}(\omega) = D_1$ and

$$\Psi(\omega) = d_0[\omega_0] + d_1[\omega_1] + e_0[\eta_0] + e_1[\eta_1] \tag{5.4}$$

for some d_0, d_1, e_0, e_1. We then have

$$h_p(D_1, D_3) = {}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \omega_{D_1} = {}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \frac{y(Q)}{x - x(Q)}\frac{dx}{y} - d_0\, {}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \omega_0 - d_1\, {}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \omega_1.$$

Using [KK22, Kay22], we compute

$$\begin{aligned}
{}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \frac{y(Q)}{x - x(Q)}\frac{dx}{y} &= 2\cdot 5 + 5^2 + 4\cdot 5^3 + 3\cdot 5^4 + 4\cdot 5^5 + 2\cdot 5^6 + O(5^7), \\
{}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \omega_0 &= 3\cdot 5 + 3\cdot 5^2 + 5^4 + 4\cdot 5^5 + O(5^7), \\
{}^{\mathrm{Vol}}\!\!\int_{w(P)}^{S} \omega_1 &= 3\cdot 5 + 4\cdot 5^2 + 3\cdot 5^3 + 4\cdot 5^4 + 2\cdot 5^6 + O(5^7).
\end{aligned}$$

In order to determine d_0 and d_1, write

$$\Psi(\omega) = c_0[\omega_0] + c_1[\omega_1] + c_2[\omega_2] + c_3[\omega_3]. \tag{5.5}$$

Comparing (5.4) and (5.5), we see that

$$\begin{aligned}
e_1 &= c_3/l_3, & e_0 &= (c_2 - e_1\cdot l_2)/k_2, \\
d_1 &= c_1 - e_0\cdot k_1 - e_1\cdot l_1, & d_0 &= c_0 - e_0\cdot k_0 - e_1\cdot l_0.
\end{aligned}$$

The cup product matrix N with respect to the basis $\mathcal{B} = \{[\omega_0], [\omega_1], [\omega_2], [\omega_3]\}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & -10/3 \\ 0 & -1 & 0 & -56 \\ -1/3 & 10/3 & 56 & 0 \end{pmatrix}$$

and the global symbols are

$$\begin{pmatrix} \langle \omega, \omega_0 \rangle \\ \langle \omega, \omega_1 \rangle \\ \langle \omega, \omega_2 \rangle \\ \langle \omega, \omega_3 \rangle \end{pmatrix} = \begin{pmatrix} 2 \cdot 5 + 5^2 + 4 \cdot 5^3 + 4 \cdot 5^4 + 3 \cdot 5^5 + 4 \cdot 5^6 + O(5^7) \\ 4 \cdot 5 + 2 \cdot 5^2 + 4 \cdot 5^3 + 4 \cdot 5^4 + 2 \cdot 5^5 + O(5^7) \\ 2 + 3 \cdot 5 + 2 \cdot 5^3 + 4 \cdot 5^4 + 4 \cdot 5^5 + O(5^7) \\ 2 \cdot 5^2 + 2 \cdot 5^3 + 3 \cdot 5^4 + 2 \cdot 5^5 + 4 \cdot 5^6 + O(5^7) \end{pmatrix}.$$

We then get that

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + 4 \cdot 5^4 + 3 \cdot 5^6 + O(5^7) \\ 2 + 4 \cdot 5 + 5^3 + 2 \cdot 5^5 + 5^6 + O(5^7) \\ 5 + 3 \cdot 5^2 + 2 \cdot 5^3 + 5^4 + 2 \cdot 5^5 + 5^6 + O(5^7) \\ 4 \cdot 5 + 2 \cdot 5^3 + 3 \cdot 5^5 + O(5^7) \end{pmatrix}$$

which implies that

$$\begin{pmatrix} e_1 \\ e_0 \\ d_1 \\ d_0 \end{pmatrix} = \begin{pmatrix} 3 \cdot 5 + 3 \cdot 5^2 + 5^5 + O(5^7) \\ 5 + 4 \cdot 5^2 + 5^3 + 4 \cdot 5^4 + 5^5 + 3 \cdot 5^6 + O(5^7) \\ 2 + 4 \cdot 5 + 3 \cdot 5^2 + 2 \cdot 5^3 + 3 \cdot 5^4 + 2 \cdot 5^5 + 5^6 + O(5^7) \\ 4 \cdot 5 + 4 \cdot 5^2 + 3 \cdot 5^3 + 4 \cdot 5^4 + 4 \cdot 5^6 + O(5^7) \end{pmatrix}.$$

Combining all of this, we obtain

$$h_p(D_1, D_3) = 5 + 3 \cdot 5^2 + 2 \cdot 5^3 + 2 \cdot 5^4 + 3 \cdot 5^5 + 4 \cdot 5^6 + O(5^7).$$

A similar computation, which we omit, gives that

$$h_p(D_4, D_2) = 4 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + 4 \cdot 5^4 + 4 \cdot 5^5 + 4 \cdot 5^6 + O(5^7).$$

We recomputed these local heights using the code [Bia23b], and recovered the same results, confirming [Bia23a, Corollary 5.32] in this example.

Finally, we see that

$$h(D_1, D_3) = \sum_q h_q(D_1, D_3) = 5 + 3 \cdot 5^2 + 2 \cdot 5^3 + 2 \cdot 5^4 + 3 \cdot 5^5 + 4 \cdot 5^6 + O(5^7),$$
$$h(D_4, D_2) = \sum_q h_q(D_4, D_2) = 4 \cdot 5 + 4 \cdot 5^2 + 3 \cdot 5^3 + 3 \cdot 5^4 + 2 \cdot 5^5 + 4 \cdot 5^6 + O(5^7),$$

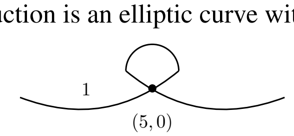
and hence get the following result, as predicted by (5.3)

$$6h(D_1, D_3) = 5 + 4 \cdot 5^2 + 5^5 + 3 \cdot 5^6 + O(5^7) = 4h(D_4, D_2).$$

Example 5.4. (Genus 2) Consider the hyperelliptic curve $C/\mathbb{Q}$ given by

$$y^2 = x^5 - 4x^4 - 48x^3 + 64x^2 + 512x + 256$$

which is obtained from the curve [LMF24, 125237.a.125237.1] by applying the change of variables $x \mapsto x/4$ and $y \mapsto y/16$. The prime $p = 7$ is a prime of bad reduction for C and the corresponding (stable) reduction is an elliptic curve with an ordinary double point:



Consider the points $P = (-4, 16)$, $Q = (-3, -5)$, $R = (0, 16)$, $S = (4, 16)$, $T = (8, 16)$, $U = (36, 7184)$ on C, and set

$$D_1 = (P) - (w(P)), \qquad D_2 = (R) - (w(R)), \qquad D_3 = (Q) - (S),$$
$$D_4 = (T) - (U), \qquad D_5 = (w(T)) - (w(U)).$$

The canonical subspace W_p^C is generated by

$$[\eta_0] = k_0[\omega_0] + k_1[\omega_1] + k_2[\omega_2] \quad \text{and} \quad [\eta_1] = l_0[\omega_0] + l_1[\omega_1] + l_2[\omega_2] + l_3[\omega_3],$$

where

$$k_0 = 7 + 6 \cdot 7^2 + 2 \cdot 7^3 + 5 \cdot 7^4 + 2 \cdot 7^5 + O(7^6),$$
$$k_1 = 2 + 6 \cdot 7 + 7^2 + 4 \cdot 7^3 + 3 \cdot 7^4 + 7^5 + O(7^6), \qquad k_2 = 1,$$
$$l_0 = 2 + 7 + 6 \cdot 7^2 + 4 \cdot 7^4 + 7^5 + O(7^6), \qquad l_2 = -12,$$
$$l_1 = 4 \cdot 7 + 4 \cdot 7^2 + 6 \cdot 7^3 + 4 \cdot 7^4 + 3 \cdot 7^5 + O(7^6), \qquad l_3 = 3.$$

Following the same steps as in Example 5.3, we find that

$$\begin{aligned} h(D_1, D_2) &= \mathbf{0} + 7 + 6 \cdot 7^2 + 2 \cdot 7^3 + 3 \cdot 7^4 + 3 \cdot 7^5 + 6 \cdot 7^6 + O(7^7), \\ &= 7 + 6 \cdot 7^2 + 2 \cdot 7^3 + 3 \cdot 7^4 + 3 \cdot 7^5 + 6 \cdot 7^6 + O(7^7), \\ h(D_3, D_4) &= \mathbf{-3 \cdot \log_p(2) + \log_p(13)} + 3 \cdot 7^3 + 2 \cdot 7^5 + 6 \cdot 7^7 + O(7^8), \\ &= 4 \cdot 7 + 7^2 + 3 \cdot 7^3 + 7^4 + 2 \cdot 7^5 + 4 \cdot 7^6 + 6 \cdot 7^7 + O(7^8), \\ h(D_3, D_5) &= \mathbf{\log_p(3) - \log_p(11)} + 2 \cdot 7^2 + 3 \cdot 7^3 + 5 \cdot 7^4 + 3 \cdot 7^5 + 2 \cdot 7^6 + O(7^7), \\ &= 3 \cdot 7 + 5 \cdot 7^2 + 3 \cdot 7^3 + 5 \cdot 7^4 + 4 \cdot 7^5 + 2 \cdot 7^6 + O(7^7)\,, \end{aligned}$$

where the bold-faced terms are the local contributions away from $p = 7$. Note that the computation of the local heights at 7 is somewhat more involved than in Example 5.3, because the computation of the relevant Vologodsky integrals requires the computation of integrals of differentials of the third kind on an elliptic curve with good reduction (reducing to the elliptic curve in the stable reduction at 7), whereas in Example 5.3, the actual Coleman integrals were all computed on $\mathbb{P}^1$.

Again, we used [Bia23b] to show that these results are consistent with [Bia23a, Corollary 5.32].

Example 5.5. (Quadratic Chabauty) According to [LMF24, 85280.d.682240.1], the genus 2 curve

$$C\colon y^2 = x^5 + x^3 - 2x + 1 \tag{5.6}$$

has good reduction away from $2, 3, 5, 41$ and its Jacobian J satisfies

$$J(\mathbb{Q}) = \langle P_1, P_2 \rangle \cong \mathbb{Z} \times \mathbb{Z}\,; \quad P_1 = [(0,1) - (1,1)],\ P_2 = [(1,-1) - \infty].$$

We now use quadratic Chabauty at the bad prime $p = 5$ as outlined in §4.1.1 to show that the set of integral points on C consists precisely of the points

$$(0, \pm 1), (1, \pm 1), (-1, \pm 1)\,. \tag{5.7}$$

If χ_v is unramified, then [BBM16, Proposition 3.3] implies that the function that maps an integral point $P \in C_v(\mathbb{Z}_v)$, to $h_v(P - \infty, P - \infty)$, normalised as in Remark 3.5, is constant on preimages of irreducible components of any proper regular model of C_v over $\mathbb{Z}_v$, and is identically 0 when there is such a model with irreducible special fibre. Hence the same is true for the function ν_v defined in (4.7) (see also [Bia23a, Lemma 7.2] and [Bia23a, Theorem 7.1], which summarises various results from [MS16]). The equation (5.6) defines a proper regular model for all $v \neq 2$, which implies that the sets in Theorem 4.5 satisfy $\Gamma_v = \{0\}$ for all $v \neq 2$. At 2, the minimal proper regular model has four components, all of genus 0. The value of ν_2 is always 0 for the component containing 0, since this holds for the local height $h_v(P-\infty, P-\infty)$ by [BBM16, Proposition 3.3]. The $\mathbb{Z}_2$-integral points $(1,1), (-1,1), (4/9, 339/4)$ map to the other three components, respectively, and we compute that ν_2 takes the values $-2\log_p(2)$ and $-\frac{8}{3}\log_p(2)$ there. Thus the set Γ in Theorem 4.5 is

$$\Gamma = \Gamma_2 = \left\{0, 2\log_p(2), \frac{8}{3}\log_p(2)\right\}.$$

We work with the bad prime $p = 5$, and we choose the subspace W_5 corresponding to the matrix $c = 0$. Since x^5+x^3-2x+1 reduces to $(x+2)^2(x^3+x^2+3x+4)$, [BS10, Figure 1] and [Bia23a, Theorem 7.1] imply that mP lies in the kernel of reduction of $J(\mathbb{Q}_5)$ for any $P \in J(\mathbb{Q}_5)$, where $m = (p-1)\#E(\mathbb{F}_5) = 4 \cdot 9 = 36$ and $E/\mathbb{F}_5$ is the elliptic curve defined by $y^2 = x^3+x^2+3x+4$. In this example, the kernel of reduction equals the subgroup H_5 by [Bia23a, Section 3]. Hence we may always use the factor 36 to compute the abelian logarithm $\mathcal{L}$ on $J(\mathbb{Q}_5)$ and the Néron function λ_5. We use this to express the global height h as a linear combination of the products $\mathcal{L}_1^2, \mathcal{L}_1\mathcal{L}_2, \mathcal{L}_2^2$ by evaluating these in the points $P_1, P_3 = [(0,1)-(1,-1)] \in J(\mathbb{Q})$, which generate a finite index subgroup of $J(\mathbb{Q})$ and which have the property that $P_1, P_3, P_1+P_3 \in J(\mathbb{Q}) \setminus \mathrm{Supp}(\Theta)$. The latter condition is not necessary if we compute the heights and the $\mathcal{L}_i$ using Vologodsky integration as in Section 3 and in [KK22, Kay22], but is required to apply the results of [Bia23a] discussed in Section 4, which is what we used for this example. We find that

$$h = a_1\mathcal{L}_1^2 - a_2\mathcal{L}_1\mathcal{L}_2 - a_3\mathcal{L}_2^2\,, \tag{5.8}$$

where

$$\begin{aligned} a_1 &= 4\cdot 5^{-1} + 1 + 4\cdot 5^3 + 5^4 + 5^6 + O(5^7),\\ a_2 &= 4\cdot 5^{-1} + 3 + 5 + 4\cdot 5^6 + O(5^7),\\ a_3 &= 1 + 4\cdot 5 + 5^3 + 4\cdot 5^4 + 3\cdot 5^5 + 4\cdot 5^6 + O(5^7)\,. \end{aligned}$$

Hence the quadratic Chabauty function ρ in Theorem 4.5 is

$$\rho := \nu_5 - a_1\mathcal{L}_1^2 - a_2\mathcal{L}_1\mathcal{L}_2 - a_3\mathcal{L}_2^2\,, \tag{5.9}$$

with a_i's as above and ν_5 defined in (4.7).

In order to solve for the set of all $P \in C(\mathbb{Z}_5)$ such that $\rho(P) \in \Gamma$, we need to expand ρ in residue discs. Since no $\mathbb{Q}_5$-point lies in the residue disc reducing to the singular point $(3,0) \in C(\mathbb{F}_5)$, we only need to consider the discs reducing to $(0,\pm 1), (1,\pm 1), (-1,\pm 1)$. These all contain at least one integral point (x_0, y_0), which we may use to parametrize them using the local parameter $z = x - x_0$. Also note that by symmetry, we only need to find the roots of ρ in these disc up to the hyperelliptic involution. In order to expand the functions ν_5 and $\mathcal{L}_i$ in the local parameter z, we need to work over the Laurent series ring $\mathbb{Q}[[z]]$ (see for instance (4.4), noting that the coefficients of the expansion of the coordinates of a parametric point $P(z)$ on C have rational coefficients). In particular, we need to perform scalar multiplication by 36 on $J(\mathbb{Q}[[z]])$. Rather than using Cantor's algorithm, which does not behave well over inexact fields, we instead perform scalar multiplication on the Kummer surface $K := J/\langle -1\rangle$. This is a quartic hypersurface that remembers much of the arithmetic on J. In particular, let $\kappa\colon J \to K \subset \mathbb{P}^3$ be the map in [FS97], then, for any $P \in J$ and $n \in \mathbb{Z}$, we can compute $\kappa(nP)$ from $\kappa(P)$ by repeatedly evaluating certain quartic forms and biquadratic forms as in [FS97] – without loss of precision. From $\kappa(nP)$, we can then find the unordered pair $\{\pm nP\}$. Since the functions appearing in ρ are even (in Grant's formal group parameters T_1, T_2), this ambiguity is not an issue for us.

After solving $\rho \in \Gamma$, we find, in addition to the known integral points (5.7), approximations of four "extra" p-adic points on C:

$$(1+2\cdot5+2\cdot5^3+2\cdot5^4+2\cdot5^5+O(5^6)\,,\;1+5+4\cdot5^2+3\cdot5^3+4\cdot5^4+5^5+O(5^6)),$$
$$(4\cdot5+2\cdot5^2+3\cdot5^3+O(5^6)\,,\;1+5+4\cdot5^2+5^3+5^4+4\cdot5^5+O(5^6)),$$
$$(4+2\cdot5+2\cdot5^2+4\cdot5^3+3\cdot5^4+2\cdot5^5+O(5^6)\,,\;4+5^2+5^3+3\cdot5^4+3\cdot5^5+O(5^6)),$$
$$(4+5+4\cdot5^2+4\cdot5^3+4\cdot5^4+O(5^6)\,,\;4+3\cdot5+2\cdot5^3+2\cdot5^5+O(5^6))$$

It remains to show that these points are not $\mathbb{Z}$-integral. To this end we use the Mordell–Weil sieve as in [BBM17]: Assuming that an extra point P is actually in $C(\mathbb{Q})$, there are coefficients $a_1, a_2 \in \mathbb{Z}$ satisfying $\iota(P) = a_1P_1 + a_2P_2$. We use the linearity of the abelian logarithms $\mathcal{L}_1, \mathcal{L}_2$, again relying on Kummer arithmetic to avoid loss of precision in arithmetic in $J(\mathbb{Q}_p)$, to approximate these putative coefficients modulo 5^6, which results in a coset of $5^6J(\mathbb{Q})$ in $J(\mathbb{Q})$ that would have to contain $\iota(P)$. We show that the image of this coset in $J(\mathbb{F}_{311})/5^6J(\mathbb{F}_{511})$ does not intersect the image of $C(\mathbb{F}_{311})$ for any extra point, which finally proves that $C(\mathbb{Z})$ consists precisely of the points in (5.7).

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