Computing Tame Splitting Densities

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Motivation

What proportion of quadratic polynomials over $\mathbb{Z}$ generate a quadratic extension which is ramified at some fixed odd prime p?

It's the same as the probability that $b^2 - 4ac$ is divisible by p an *odd* number of times. So, work p-adically to find that the answer is

$$\frac{p}{p^2 + p + 1}$$

.

Notice three things.

1. $\frac{p}{p^2+p+1}$ is rational.
2. $\frac{p}{p^2+p+1}$ is a rational function in terms of p.
3. $\frac{p}{p^2+p+1} = \frac{1/p}{(1/p)^2+(1/p)+1}$.

General Question

What about higher degrees? What's the probability that a degree d polynomial generates an extension totally ramified (or inert, or totally split, or somewhere in the middle) at p?

Theorem (Del Corso, Dvornicich (2000))

They're rational functions, at least for p tame.

Conjecture (Bhargava, Cremona, Fisher, Gajovic (2022))

They're rational function f in terms of p, satisfying $f(p) = f(1/p)$.

This was proven in 2023 by G, Wei, Y. as a special case of a more general theorem. I calculate this rational function concretely with code uploaded on my Github (link is on my website). Associated paper is also uploaded to arXiv.

Explicit Computations

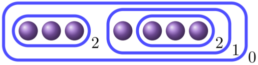
Example

The proportion of degree 5 polynomials generating an extension totally ramified over p is

$$\frac{p^{10}+p^4}{p^{14}+p^{13}+2\,p^{12}+p^{11}+2\,p^{10}+p^9+2\,p^8+p^7+2\,p^6+p^5+2\,p^4+p^3+2\,p^2+p+1}.$$

Theorem (Dokchitser, Dokchitser, Maistret, Morgan (2018))

Given $f \in \mathbb{Z}_p[x]$, the distances between the roots of f give much information about the hyperelliptic curve $y^2 = f(x)$.



Work in progress: Find the density of polynomials with specified cluster diagrams.