5-dimensional compatible systems and the Tate conjecture for elliptic surfaces

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Irreducibility in Compatible Systems

Let

$$(\rho_\ell\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell))_\ell$$

Abelian variety
$A/\mathbb{Q} \rightsquigarrow (T_\ell(A) \otimes \overline{\mathbb{Q}}_\ell)_\ell$

be a compatible system of ℓ-adic Galois representations.

- Completely determined by the polynomials $\det(1 - \rho_\ell(\mathrm{Frob}_p)T)$ for $p \notin S \cup \{\ell\}$.
- These polynomials are *independent of* ℓ.

For elliptic curves:
$1 - a_p T + pT^2$

Irreducibility Conjecture

If ρ_ℓ is irreducible for one ℓ, then ρ_ℓ is irreducible for every ℓ.

Proven by Faltings for abelian varieties. Proven by Hui (2023) when $n \leq 4$ with minor hypotheses.

Theorem (Duan–Wang–W.)

For 5-dimensional compatible systems (with very minor hypotheses):
If ρ_ℓ is irreducible for one ℓ, then ρ_ℓ is irreducible for all but finitely many ℓ.

The Tate conjecture for elliptic surfaces

$X/\mathbb{Q}$ a smooth projective variety.

$$H^2_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1))^{ss} = (\mathrm{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}_\ell) \oplus \mathrm{Tran}_\ell(X).$$

Codimension-1 ℓ-adic Tate conjecture

$\mathrm{Tran}_\ell(X)$ does not contain a copy of the trivial representation.

Theorem (Duan–Wang–W.)

- Let $X_0\colon y^2 + (t+3)xy + y = x^3$.
- $\mathcal{S}$ = set of general, degree 3, genus 2 branched multiplicative covers of X_0.
- For each $X \in \mathcal{S}$, $(\mathrm{Tran}_\ell(X))_\ell$ is a 5-dimensional compatible system.

For all but finitely many ℓ, for all but finitely many $X \in \mathcal{S}$, **the codimension-1 ℓ-adic Tate conjecture holds for X**.