Reciprocity obstructions in continued fraction semigroups

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Zaremba's conjecture

$$[a_0; a_1, a_2, \ldots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

Conjecture (Zaremba's Conjecture)

Consider all rational numbers q which have a continued fraction built from the alphabet $\{1, 2, 3, 4, 5\}$. Then every positive integer appears as a denominator of such a q.

Main result

Theorem (R.-Stange)

Consider all rational numbers q which have a continued fraction of the form

$$q = [0; a_1, a_2, \ldots, a_k, b, 1, 2]$$

where $a_k \in \{4, 8, 12, 16, \ldots\} = 4\mathbb{Z}^+$ and $b \in \mathbb{Z}^+$. Then no denominator of q is a perfect square.

Squares are not ruled out by counting or congruence.

We computed up to 2×10^{13}, and the last missing non-square is $7968219670470 \approx 7.9 \cdot 10^{12}$.

This disproves a generalization of Zaremba's conjecture made by Kontorovich.