3-Torsion Subgroups of Genus 3 Curves

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Torsion Points of Jacobians

Let C be a nice curve over $\mathbb{Q}$. Given $C(\mathbb{Q})$, we can easily construct points on its Jacobian J.

Can we compute points on J without no prior knowledge of $C(\mathbb{Q})$?

Theorem (L.,Rawson 24)

If C is a hyperelliptic curve of genus 3 or a plane quartic with a rational flex, then $J[3] \cong (\mathbb{Z}/3\mathbb{Z})^6$ is computable.

Algorithm

Input: A model of the curve (and the flex in the quartic case)
Output: A degree 728 polynomial parametrising 3-torsion points.

1. Derive a system of equations whose solutions correspond to 3-torsion points.
2. Approximate solutions of our system (Newton-Raphson, Homotopy Continuation)
3. Use our approximations to compute the 3-torsion polynomial (LLL, continued fractions)

An Example

Theorem (L. 2022)

Suppose C has affine model $y^2 = f(x) = x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, where $a_i \in K$. Then any non-zero 3-torsion point of J is the form $\left[\frac{1}{3}div(h)\right]$ where $h = y(x + \alpha_1) + \alpha_2x^4 + \alpha_3x^3 + \alpha_4x^2 + \alpha_5x + \alpha_6$, with $\alpha_i \in \overline{K}$ satisfying

$$f(x)(x + \alpha_1)^2 + \alpha_7\left(x^3 + \alpha_8x^2 + \alpha_9x + \alpha_{10}\right)^3 = \left(\alpha_2x^4 + \alpha_3x^3 + \alpha_4x^2 + \alpha_5x + \alpha_6\right)^2$$

$y^2 + (x^4 + x^3 + x^2 + 1)y = x^3 + x^2 + x$

$x^{728} + 201212x^{726} + 17405182518x^{724} + 786349512797185x^{722} + 1951909156357720631 7x^{720} + 2843861417344768362527 61x^{718} +$

$22684941726577220163318658 05x^{716} + \ldots + -17950611908110863313962317303788517366065966067194925320938142496211454145839 51$

445054107476768416544571388356105262687664424089205133843376822230378801756559688171261673707236099820914244701624153246218646383507

0818483929868959210458154385940321326469096577188797466940053427094320754456312