Census of genus 6 curves over $\mathbf{F}_2$

Joint work with Kiran S. Kedlaya and Jun Bo Lau

Steve (Yongyuan) Huang[1]

[1]Department of Mathematics
University of California San Diego

Sixteenth Algorithmic Number Theory Symposium
MIT, Cambridge, MA
July 16, 2024

Motivation and Problem Statement

Motivating Question

There are 164,937 isogeny classes of abelian varieties of dimension 6 over $\mathbf{F}_2$. How many of them contain the Jacobian of curves of genus 6?

Answer (H.–Kedlaya–Lau): 38,327.
Let $\mathcal{M}_g$ denote the moduli space of smooth curves of genus g.
As sets,

$$\mathcal{M}_g(k) \longleftrightarrow \{\text{isomorphism classes of curves of genus } g \text{ over } k\}.$$

For finite fields, $\#\mathcal{M}_g(k)$ is finite; Bergstrom–Canning–Petersen–Schmitt has obtained the polynomial point count formula

$$\#\mathcal{M}_6(\mathbf{F}_q) = q^{15} + q^{14} + 2q^{13} + q^{12} - q^{10} + q^3 - 1 \implies \#\mathcal{M}_6(\mathbf{F}_2) = 68,615.$$

Problem Statement

Enumerate $\mathcal{M}_6(\mathbf{F}_2)$, i.e. find one curve representing each isomorphism class and compute the order of its automorphism group over $\mathbf{F}_2$.

Methodology

We first enumerate a covering set for the isomorphism classes of curves for each stratum in the Brill–Noether stratification of $\mathcal{M}_6$, and then filtering redundancies using functionalities in MAGMA.

Theorem (Enriques–Petri, Mukai+ϵ)

Let C be a curve of genus 6 over a finite field k. Then C is exactly one of the following:

1. *Hyperelliptic.*
2. *Bielliptic.*
3. *Smooth plane quintic in $\mathbf{P}^2_k$.*
4. *Trigonal of Maroni invariant 0: a (3,4) in $\mathbf{P}^1_k \times_k \mathbf{P}^1_k$.*
5. *Trigonal of Maroni invariant 2: a $(2,1) \cap (1,3)$ in $\mathbf{P}^1_k \times_k \mathbf{P}^2_k$.*
6. *Brill-Noether-general: a $(1)^4 \cap (2) \cap \mathrm{Gr}(2,5)$ in $\mathbf{P}^9_k$.*