Abelian extensions arising from elliptic curves with complex multiplication

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ANTS XVI,
July 15-19, 2024

Background and Motivation

Let E be an elliptic curve defined over a number field F.

- Let $N \geq 2$ and $E[N] = E(\overline{F})[N]$ be the N-torsion subgroup of $E(\overline{F})$.
- $F(E[N])$ is the field of definition of the coordinates of points in $E[N]$.
- $F(E[N])/F$ is a Galois extension.

Q1: When is $F(E[N])/F$ an abelian extension?

Theorem (H. and Lozano-Robledo, 2023)

Let E/F be an elliptic curve with CM, where $F = \mathbb{Q}(j(E))$. Then $F(E[N])/F$ is only abelian for $N = 2$, 3, or 4.

Q2: What is the maximal abelian extension contained in $F(E[N])/F$?

Main Theorem

Theorem (H., 2024)

Let $E/\mathbb{Q}$ be an elliptic curve with CM by an order $\mathcal{O}_{K,f}$ of K, $f \geq 1$.

(1) *If p is any prime such that $p \nmid \Delta_K f^2$, then*

$$\mathbb{Q}(E[p^n]) \cap \mathbb{Q}^{ab} = K(\zeta_{p^n}).$$

(2) *If $p > 2$ prime such that $p \mid \Delta_K f^2$, then*

$$\mathbb{Q}(E[p^n]) \cap \mathbb{Q}^{ab} = \begin{cases} \mathbb{Q}(\zeta_{p^n}, \sqrt{\alpha}) & \text{if } E \text{ is a quadratic twist by } \alpha, \\ \mathbb{Q}(\zeta_{p^n}) & \text{if } E \text{ is a simplest model.} \end{cases}$$

(3) *Let $p = 2$ such that $2 \mid \Delta_K f^2$.*

- *If $\Delta_K f^2 = -12$ or -28, then* $\mathbb{Q}(E[2^n]) \cap \mathbb{Q}^{ab} = K(\zeta_{2^{n+1}})$.
- *If $\Delta_K f^2 = -4, -8,$ or -16, then*

$$\mathbb{Q}(E[2^n]) \cap \mathbb{Q}^{ab} = \begin{cases} \mathbb{Q}(\zeta_{2^{n+1}}, \sqrt{\alpha}) & \text{if } E \text{ is a quadratic twist by } \alpha, \\ \mathbb{Q}(\zeta_{2^{n+1}}) & \text{if } E \text{ is a simplest model.} \end{cases}$$