Dehn Invariant Zero Tetrahedra

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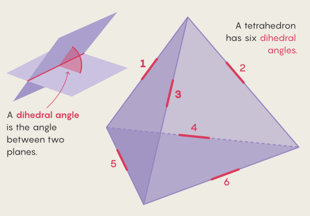
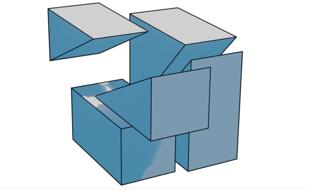


(a) Six dihedral angles (Quanta Magazine)



(b) Scissors Congruence (Numberphile)

For edge lengths ℓ_e and dihedral angles α_e

$$\text{Dehn Invariant } := \sum_{\text{edges } e} \ell_e \otimes \alpha_e \in \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\mathbb{Q}\pi).$$

Theorem (Dehn-Sydler; Hilbert's 3rd Problem)

Two polyhedra are scissors-congruent if and only if their volumes and Dehn invariants are both equal.

Given a tetrahedron $\mathcal{T}$, define $V_{\mathcal{T}} := \text{Span}_{\mathbb{Q}}(\{\alpha_e\}_e) \subseteq \mathbb{R}/\mathbb{Q}\pi$.

Rational Tetrahedra

The case $\dim(V_{\mathcal{T}}) = 0$ corresponds to tetrahedra whose dihedral angles α_e all lie in $\mathbb{Q}\pi$. These tetrahedra were fully classified by Kedlaya, Kolpakov, Poonen, and Rubinstein.

We are interested in the maximal possible $\dim(V_{\mathcal{T}}) = 5$.

Theorem (C.-Sun)

Up to scaling, there exist (effectively) finitely many tetrahedra $\mathcal{T}$ of Dehn invariant zero for which $\dim(V_{\mathcal{T}}) = 5$.

Lemma

In the case $\dim(V_{\mathcal{T}}) = 5$*, the tetrahedron can be scaled so* $\ell_e \in \mathbb{Z}$ *and* $\sum_e \ell_e \alpha_e \in \mathbb{Q}\pi$*.*

Let $z_e = \exp(i\alpha_e)$. One can show that $z_e^2 \in \mathbb{Q}(\sqrt{-2D})$ where $D \in \mathbb{Z}$ is the *Cayley-Menger* determinant. If $\dim(V_{\mathcal{T}}) = 5$, then

$$\mathcal{T} \text{ has Dehn invariant zero} \iff \prod z_e^{\ell_e} \text{ is a root of unity}$$
$$\iff \prod z_e^{2\ell_e} \text{ is a root of unity}$$

But the roots of unity in quadratic fields are known!
Some other aspects of the problem:

1. Modulo p arguments for roots of unity
2. Computing the dimension $\dim(V_{\mathcal{T}})$
3. Classifying by symmetries in ℓ_e

THANK YOU!