On the vanishing of twisted L-functions of elliptic curves over function fields

Antoine Comeau-Lapointe, Concordia University
with Chantal David, Matilde Lalín, and Wanlin Li

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The L-function associated to an elliptic curve

Given an elliptic curve defined over $\mathbb{F}_q[t]$

$$E : y^2 = x^3 + a(t)x + b(t),$$

its L-function is defined as

$$L(E, u) = \prod_{P \nmid N_E} (1 - a_P u^{\deg(P)} + q^{\deg(P)} u^{2\deg(P)})^{-1} \prod_{P | N_E} (1 - a_P u^{\deg(P)})^{-1},$$

where $a_P = q^{\deg(P)} + 1 - \#E/P$ for irreducible polynomials $P \nmid N_E$, and $a_P = 0, 1$, or -1 depending on the type of bad reduction when $P \mid N_E$.

Given a Dirichlet character χ, the twisted L-function of E by χ is defined as

$$L(E, \chi, u) = \prod_{P \nmid N_E} (1 - a_P \chi(P) u^{\deg(P)} + q^{\deg(P)} \chi^2(P) u^{2\deg(P)})^{-1} \times \prod_{P | N_E} (1 - a_P \chi(P) u^{\deg(P)})^{-1}.$$

Results of Li and Donepudi-Li

Conjecture (Chowla,1965): $L(\chi, 1/2) \neq 0$ for any Dirichlet character over $\mathbb{Q}$.

Theorem (Li, 2018): There exists infinitely many quadratic characters such that $L(\chi, q^{-1/2}) = 0$ over $\mathbb{F}_q[t]$.

Theorem (Donepudi-Li, 2020): For ℓ prime, there exists infinitely many characters of order ℓ such that $L(\chi, q^{-1/2}) = 0$ over $\mathbb{F}_q[t]$ for specific values of q.

Results of Li and Donepudi-Li

Given a Dirichlet character χ_0 over $\mathbb{F}_q[t]$, we can construct infinitely many characters χ' such that

$$L(\chi_0, u) \mid L(\chi', u).$$

When $q \equiv 1 \bmod \ell$, χ_0 is associated to the zeta function of the curve

$$y^\ell = F(t).$$

Given $h(t) \in \mathbb{F}_q(t)$, we can construct such χ' using the map $t \mapsto h(t)$

$$y^\ell = F(h(t)).$$

Result of Bary-Soroker and Meisner

When $q \not\equiv 1 \bmod \ell$, a Dirichlet character χ of order ℓ is associated to

$$y^\ell + a_{\ell-1}(t)y^{\ell-1} + \cdots + a_1(t)y + a_0(t) = 0.$$

L-functions of constant elliptic curves

$$E : y^2 = x^3 + a(t)x + b(t)$$

An elliptic curve is constant when $a(t), b(t) \in \mathbb{F}_q \subset \mathbb{F}_q[t]$. We denote its Frobenius eigenvalues by α_1 and α_2.

We have

$$L(\chi, u) = \prod_{1 \leq j \leq 2g/(\ell-1)} (1 - \gamma_j u)$$

where g is the genus of χ. Then

$$L(E, \chi, u) = \prod_{\substack{1 \leq i \leq 2 \\ 1 \leq j \leq 2g/(\ell-1)}} (1 - \alpha_i \gamma_j u).$$

If we can find χ_0 such that some γ_j equals α_1 or α_2, we get infinitely many χ such that $L(E, \chi, q^{-1}) = 0$.

Examples

Let χ_0 be a character associated to the following curve over $\mathbb{F}_{13}$

$$y^7 + (6t^4 + 6t^3 + 6t^2 + 12t + 1)y^5 + (t^8 + 2t^7 + 3t^6 + 6t^5 + t^4 + 5t + 4)y^3 + (6t^{12} + 5t^{11} + 10t^{10} + 7t^8 + 2t^7 + 3t^6 + 9t^5 + 3t^4 + 2t^3 + 6t^2 + t + 4)y + 11t^{14} + 6t^{13} + 12t^{12} + 10t^{11} + 5t^{10} + 8t^9 + 6t^8 + 2t^7 + 2t^6 + 10t^5 + 7t^4 + 12t^3 + 3t^2 + 3t + 9 = 0.$$

We have $L(\chi_0, u) = 1 + 13u^2$.

We believe that when $q \equiv -1 \bmod \ell$, there is always a character with L-function equal to $1 + qu^2$.

Conjecture over $\mathbb{Q}$

Conjecture (David–Fearnley–Kisilevsky, 2007; Mazur–Rubin, 2019): For any $E/\mathbb{Q}$ and $\ell > 5$, there are only finitely many χ such that $L(E, \chi, 1) = 0$.

Our result do not work for non-constant elliptic curves because the map $t \mapsto h(t)$ changes the elliptic curve

$$E' : y^2 = x^3 + a(h(t))x + b(h(t)).$$

Numerical Computations

Source code available at **github.com/AntoineComeau/Lfuncff**

We compute the coefficients of the L-function directly.

$$L(E, \chi, u) = \sum_{n=0}^{M} \left(\sum_{f \in \mathcal{M}_n} a_f \chi(f) \right) u^n$$

where $\mathcal{M}_n$ is the set of monic polynomials of degree n in $\mathbb{F}_p[t]$.

Data

twist order	p	n_p	deg conductor d	rank 0	rank 1	rank 2
3	5	2	2	6	4	0
			4	205	32	3
			6	5784	260	16
			8	302640	116	4
	7	1	1	5	0	0
			2	37	4	0
			3	324	37	1
			4	2935	73	0

Table 1: Twists of order 3 for the Legendre curve $y^2 = x(x-1)(x-t)$.

Data

twist order	p	n_p	deg conductor d	rank 0	rank 1
7	5	6	6	2560	20
	11	3	3	440	0
	13	2	2	72	6
			4	24984	132
	29	1	1	24	0
			2	2046	16
	41	2	2	800	20

Table 2: Twists of order 7 for the curve $y^2 = (x-1)(x-2t^2-1)(x-t^2)$.

We did not find any character of order > 7 with rank greater than zero.

Thank you for your attention!