Definite orthogonal modular forms: Computations, Excursions and Discoveries

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presents

FUN WITH L-FUNCTIONS

Fun with L-functions

Lattice Λ **magic** $\rightsquigarrow$ **orthogonal modular forms** ϕ_i.

Example ($n = 4, D = 37^2$)

For Λ with Gram matrix $\begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 4 & 1 & 2 \\ 1 & 1 & 10 & 1 \\ 1 & 2 & 1 & 20 \end{pmatrix}$ we get

$$L_p(\phi_1, T) = (1 - T)(1 - pT)^2(1 - p^2 T)$$
$$L_p(\phi_2, T) = (1 - (a_p^2 - 2p)T + p^2 T^2)(1 - pT)^2$$
$$L_p(\phi_3, T) = (1 - (b_p^2 - 2p)T + p^2 T^2)(1 - pT)^2$$
$$L_p(\phi_4, T) = (1 - pb_p T + p^3 T^2)(1 - b_p T + pT^2)$$

where a_p, b_p are coefficients of 37.2.a.a and 37.2.a.b.

time ($p < 100$): 109.15s (2.3GHz 8-Core Intel Core i9)

Symmetric Square L-functions

Lattice Λ **magic** $\rightsquigarrow$ **orthogonal modular forms** ϕ_i.

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For Λ with Gram matrix $\begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 4 & 1 & 2 \\ 1 & 1 & 10 & 1 \\ 1 & 2 & 1 & 20 \end{pmatrix}$ we get

$$L_p(\phi_1, T) = (1 - T)(1 - pT)^2(1 - p^2T)$$
$$L_p(\phi_2, T) = (1 - pT)L_p(\mathrm{Sym}^2(f), T)$$
$$L_p(\phi_3, T) = (1 - pT)L_p(\mathrm{Sym}^2(g), T)$$
$$L_p(\phi_4, T) = L_p(E \otimes f, T)$$

where $f, g \in S_2(37)$ are 37.2.a.a and 37.2.a.b.

time ($p < 20$): 1.75s (2.3GHz 8-Core Intel Core i9)
time ($p < 100$): 109.15s (2.3GHz 8-Core Intel Core i9)

Fun with L-functions (Rank 6)

Example ($n = 6, D = 39$)

For $\Lambda = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$ we find a ϕ with

$$L_p(\phi_1, T) = (1 - \chi(p)p^2 T) \prod_{i=0}^{4} (1 - p^i T)$$

$$L_p(\phi_2, T) = L_p(\chi \otimes \mathrm{Sym}^2(f), T) \prod_{i=1}^{3} (1 - p^i T)$$

where $f \in S_3(39, \chi)$ is 39.3.d.c and $\chi = \chi_{-39}$.

time ($p < 20$): 4281.50s (2.3GHz 8-Core Intel Core i9)

Fun with L-functions (Rank 8)

Example ($n = 8, D = 53$)

For Λ of rank 8 and discriminant 53 we get

$$L_p(\phi, T) = (1 - \chi(p)p^3 T)\prod_{i=2}^{4}(1 - p^i T)G_p(T)$$

where $G_p(T)$ is irreducible with $\deg G_p(T) = 4$ and $\chi = \chi_{53}$.

time ($p < 10$): 354957.51 (2.3GHz 8-Core Intel Core i9)

- How do we prove the identities in the first two examples?
- Where does $G_p(T)$ come from?

Genus and Class set

The **genus** of $\Lambda \subseteq V$ is

$$\mathrm{gen}(\Lambda) := \{\Pi \subseteq V : \Lambda_p \simeq \Pi_p \text{ for all } p\}.$$

The **class set** $\mathrm{cls}(\Lambda) = \mathrm{gen}(\Lambda)/\simeq$ is the set of (global) isometry classes in $\mathrm{gen}(\Lambda)$.

Neighbors

Kneser's theory of p^k-neighbors gives an effective method to compute the class set; it also gives a Hecke action!
Let $p \nmid \operatorname{disc}(\Lambda)$ be a prime; $p = 2$ is OK.
We say that an integral lattice $\Pi \subseteq V$ is a **p^k-neighbor** of Λ, and write $\Pi \sim_{p^k} \Lambda$ if

$$\Lambda/(\Lambda \cap \Pi) \simeq (\mathbb{Z}/p\mathbb{Z})^k \simeq \Pi/(\Lambda \cap \Pi),$$

If $\Lambda \sim_{p^k} \Pi$ then $\Pi \in \operatorname{gen}(\Lambda)$.
Moreover, there exists S such that every $[\Pi] \in \operatorname{cls}(\Lambda)$ is an **iterated S-neighbor** of Λ.

$$\Lambda \sim_{p_1} \Lambda_1 \sim_{p_2} \cdots \sim_{p_r} \Lambda_r \simeq \Pi$$

with $p_i \in S$. Typically may take $S = \{p\}$.

Example - Computing the class set

Let

$$\Lambda = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 6 \end{pmatrix}$$

Thus $\operatorname{disc}(\Lambda) = 29$. We have $\# \operatorname{cls}(\Lambda) = 2$, with the nontrivial class represented by the 2-neighbor

$$\Lambda' = \frac{1}{2}\mathbb{Z}(e_2 + e_4) + 2\mathbb{Z}e_3 + \mathbb{Z}e_1 + \mathbb{Z}e_4.$$

Orthogonal modular forms

The space of **orthogonal modular forms** of level Λ is

$$M(\Lambda) := \{\phi : \mathrm{cls}(\Lambda) \to \mathbb{C}\} \simeq \mathbb{C}^{h(\Lambda)}$$

For $p \nmid \mathrm{disc}(\Lambda)$ define the **Hecke operator**

$$\begin{aligned} T_{p,k} : M(\Lambda) &\to M(\Lambda) \\ \phi &\mapsto \left([\Lambda'] \mapsto \sum_{\Pi' \sim_{p^k} \Lambda'} \phi([\Pi']) \right) \end{aligned}$$

The Hecke operators commute and are self-adjoint, hence there is a basis of simultaneous eigenvectors - **eigenforms**. (Gross, 1999)

Example - Hecke action

Let Λ be as before with discriminant 29. By checking isometry we compute w.r.t. basis $[\Lambda'], [\Lambda]$

$$[T_2] = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}, [T_3] = \begin{pmatrix} 4 & 3 \\ 6 & 7 \end{pmatrix}, [T_5] = \begin{pmatrix} 18 & 9 \\ 18 & 27 \end{pmatrix}, \ldots$$

The constant function $\phi_1 = [\Lambda] + [\Lambda']$ is an **Eisenstein series** with $T_p(\phi_1) = (p^2 + (1 + \chi_{29}(p)) + 1)\phi_1$. Another eigenvector is $\phi_2 = [\Lambda] - 2[\Lambda']$, with $T_p(\phi_2) = \lambda_p \phi_2$

$$\lambda_2 = -1, \lambda_3 = 1, \lambda_5 = 9, \lambda_7 = 4, \lambda_{11} = 17, \ldots$$

We match them with the **Hilbert modular form** labeled 2.2.29.1-1.1-a in the LMFDB.

Back to L-functions

Letting $D^* = (-1)^{\frac{n}{2}} D$ there is a natural family of theta maps:

$$\theta^{(g)} : M(\Lambda) \to M_{\frac{n}{2}}(\Gamma_0^{(g)}(D), \chi_{D^*}).$$

Theorem (A., Fretwell, Ingalls, Logan, Secord, and Voight (2022), consequence of Rallis (1982))

If n is even, ϕ is an eigenform and $f = \theta^{(g)}(\phi) \neq 0$ with $2g < n$:

$$L(\phi, s) = L\left(\chi_{D^*} \otimes f, \mathit{std}, s - \left(\frac{n}{2} - 1\right)\right) \prod_{i=g-\left(\frac{n}{2}-1\right)}^{\left(\frac{n}{2}-1\right)-g} \zeta\left(s + i - \left(\frac{n}{2} - 1\right)\right).$$

If $g = 1$, then obtain $L(\chi_D \otimes \mathrm{Sym}^2(f), s)$ and zeta factors so

$$\lambda_{p,1} = a_p^2 - \chi_{D^*}(p) p^{\frac{n}{2}-1} + p\left(\frac{p^{n-3} - 1}{p - 1}\right)$$

where a_p are the eigenvalues of f.

Application

Theorem (A., Fretwell, Ingalls, Logan, Secord, and Voight (2022))

Let $\Lambda = A_6 \oplus A_2$. There are 3 eigenforms in $M(\Lambda)$ with eigenvalues

$$\frac{p^7-1}{p-1}+\chi(p)p^3,\ \frac{p(p^5-1)}{p-1}+a_p^2-\chi(p)p^3,\ \frac{p(p^5-1)}{p-1}+b_p^2-\chi(p)p^3$$

for T_p with $p \neq 3, 7$, where $\chi = \chi_{21}$, and a_p, b_p are the eigenvalues of 21.4.c.a and 21.4.c.b, respectively.
Equivalently, there are explicit $A_1, A_2, A_3 \in M_3(\mathbb{Q})$ such that

$$T_p = \lambda_p^{(1)} A_1 + \lambda_p^{(2)} A_2 + \lambda_p^{(3)} A_3.$$

Higher depth

The **depth** d_ϕ of $\phi \in M(\Lambda)$ is the smallest integer such that $\theta^{(d_\phi)}(\phi) \neq 0$.

Example ($r = 6, D = 75$)

For $\Lambda = A_4 \oplus \Lambda_{15}$ there is a ϕ such that $d_\phi = 2$.
It **appears** that

$$\lambda_{p,1} = (p+1)a_p + p^2(1 + \chi_{-3}(p))$$

with a_p coming from $f \in S_4(5)$ (5.4.a.a).

It turns out that $\theta^{(2)}(\phi)$ is the Saito-Kurokawa lift of f so that

$$L(\phi, s) = L(\chi_{-3}, s-2)L(\chi_{-3} \otimes f, s)L(\chi_{-3} \otimes f, s-1)\zeta(s-2)$$

In general, Ikeda lifts write $L(\phi, s)$ as a product of GL_1 and GL_2 L-functions, i.e. $\lambda_{p,k}$ has a formula in terms of modular forms and Dirichlet characters.

Back to rank 8, disc 53

Recall we have

$$L_p(\phi, T) = (1 - \chi(p)p^3 T) \prod_{i=2}^{4} (1 - p^i T) G_p(T)$$

where $G_p(T)$ is irreducible with $\deg G_p(T) = 4$ and $\chi = \chi_{53}$.
Moreover, $F = \theta^{(2)}(\phi) \neq 0$, so that

$$(1 - \chi(p)p^3 T) G_p(T) = L_p(\chi \otimes F, \mathrm{std}, p^3 T).$$

Since $G_p(T)$ is irreducible, F is a non-lift Siegel modular form and

$$\lambda_{p,1} = b_{1,p^2} + p^3 + p^2 \left(\frac{p^3 - 1}{p - 1} \right)$$

where b_{1,p^2} is the T_{1,p^2} eigenvalue of F.

Eisenstein Congruences

Example (Chenevier and Lannes (2019))

Consider $\Lambda = E_8 \oplus E_8$. Then $\mathrm{cls}(\Lambda) = \{[E_8 \oplus E_8], [E_{16}]\}$ and we compute

$$T_{2,1} = \begin{pmatrix} 20025 & 18225 \\ 12870 & 14670 \end{pmatrix}$$

The constant function $\phi_1 = [E_8 \oplus E_8] + [E_{16}]$ is an **Eisenstein series** with $T_p(\phi_1) = \left(p^7 + \frac{p^{15}-1}{p-1}\right)\phi_1$. Another eigenvector is $\phi_2 = 405[E_8 \oplus E_8] - 286[E_{16}]$, with $T_p(\phi_2) = \lambda_p \phi_2$ where

$$\lambda_p = \tau(p)\left(\frac{p^4-1}{p-1}\right) + p^7 + p^4\left(\frac{p^7-1}{p-1}\right)$$

Since $286\phi_1 + \phi_2 \equiv 0 \bmod 691$, for all p we have

$$\lambda_p \equiv \left(p^7 + \frac{p^{15}-1}{p-1}\right) \bmod 691 \Rightarrow \tau(p) \equiv 1 + p^{11} \bmod 691$$

Back to rank 8, disc 53 (yes, again!)

Example ($r = 8, D = 53$)

We find ϕ_1 of depth $d_{\phi_1} = 1$ and ϕ_2 of depth $d_{\phi_2} = 2$ such that

$$273\phi_1 + \phi_2 \equiv 0 \bmod \mathfrak{q}$$

with $\mathfrak{q} \mid 397$. This implies $\lambda_{p,k} \equiv \mu_{p,k} \bmod \mathfrak{q}$ for all p, k.

$$b_{1,p^2} + p^3 + p^2\left(\frac{p^3-1}{p-1}\right) \equiv a_p^2 - \chi(p)p^3 + p\left(\frac{p^5-1}{p-1}\right) \bmod \mathfrak{q}'$$

$$b_{1,p^2} \equiv a_p^2 - (1+\chi(p))p^3 + p + p^5 \bmod \mathfrak{q}'$$

Why 397?

It **appears** that

$$\mathrm{Nm}\left(\frac{L(\mathrm{Sym}^2(f),1)}{\pi^2 L(\mathrm{Sym}^2(f),3)}\right)=\frac{24250736770795028}{2197125}\equiv 0 \bmod 397,$$

suggesting that $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(\mathrm{Sym}^2(f),6))>0$

How did we know?

Conjectural genus 2 congruences of Mizumoto-Kurokawa type:

$$b_p \equiv a_p(1 + p^{k-2}) \bmod \mathfrak{q},$$

with $\mathrm{ord}_{\mathfrak{q}}(L_{\mathrm{alg}}(\mathrm{Sym}^2(f), j + 2k - 2)) > 0$.

We see the "$\Lambda^2 - \mathrm{triv}$" of a non-trivial character version! (F contributes via "standard" GL_5 L-function, not the spinor one).

Our data allowed us to conjecture new congruences of the shape:

$$b_{1,p^2} \equiv a_p^2 - \chi(p)p^{j+k-1} - p^{j+2k-5} + p^{j+2k-3} + p^{j+1} \bmod \mathfrak{q}',$$

with χ quadratic, $f \in S_{j+k}(\Gamma_0(N), \chi)$, $F \in S_{j,k}(\Gamma_0^{(2)}(N), \chi)$ and $\mathfrak{q}$ as above.

Algorithmic aspects

- Main workhorse - computing $T_{p^k}([\Lambda])$.
  - Naive complexity - $O(hp^{k(n-k-1)})$ isometry tests.
  - Theoretically, $O(p^{k(n-k-1)})$ using a canonical form. (Sikirić, Haensch, Voight, and van Woerden, 2020)
  - In practice, better to cache via theta series.
  - Time/Memory Trade-Off - use orbits under $\mathrm{O}(\Lambda)$.
- Computing $L_p(f)$ is dominated by $O(hp^{n(n-2)/4})$.
- Genus enumeration
  - Computing $\mathrm{O}(\Lambda)$ is done by enumerating short vectors.
  - Problem when $\Lambda' \subseteq \Lambda$ has $\# \mathrm{O}(\Lambda')$ large.
  - Solved by directly finding root sublattices.

Thank you! Questions?

Magma code is available at
https://github.com/assaferan/ModFrmAlg.

```
> AttachSpec("ModFrmAlg.spec");
> Q := SymmetricMatrix([2,0,2,0,1,34,1,0,0,34]);
> M := OrthogonalModularForms(Q);
> vs := HeckeEigenforms(M);
> assert exists(v){v : v in vs |
                     HeckeEigenvalue(v,2) eq -1};
> B := 20;
> evs := HeckeEigensystem(v,1 : Precision := B);
> lpolys := [LPolynomial(v, p) : p in PrimesUpTo(B)];
> Theta1(v);
```

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Freitag, Eberhard. 1991. *Singular modular forms and theta relations*, Lecture Notes in Mathematics, vol. 1487, Springer-Verlag, Berlin.

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Gross, Benedict H. 1999. *Algebraic modular forms*, Israel J. Math. **113**, 61–93, DOI 10.1007/BF02780173. MR1729443

Rallis, Stephen. 1982. *Langlands' functoriality and the Weil representation*, Amer. J. Math. **104**, no. 3, 469–515, DOI 10.2307/2374151.