An Algorithm for Ennola's Second Theorem and Counting Smooth Numbers in Practice

Chloe Makdad
Mathematics, Statistics, and Actuarial Science Department
Computer Science and Software Engineering Department
Butler University, Indianapolis IN, USA
cmakdad@butler.edu

Jonathan P. Sorenson
Computer Science and Software Engineering Department
Butler University, Indianapolis IN, USA
jsorenso@butler.edu

August 3, 2022

ANTS Poster Abstract

Let $\Psi(x, y)$ count the number of positive integers $n \leq x$ such that every prime divisor of n is at most y.

In this poster, we address the practical problem where, given x, y, what is the best way to estimate $\Psi(x, y)$? To address this problem, we present one new algorithm, based on Ennola's second theorem [1], one improvement of an existing algorithm, shaving a factor proportional to $\log \log x$ off the running time of Algorithm HT [2] in practice, and an empirical study of five algorithms that estimate $\Psi(x, y)$, whereby we have advice on what algorithm to use given x, y as input.

Ennola's first theorem [1] has been quoted in the literature, for example, in [4, §5.2]. The second, more detailed theorem, seems to have avoided notice. We present an algorithm based on the second theorem that takes $O(y \log y)$ time to compute $\Psi(x, y)$, with $O(y^2/\log y)$ precomputation time that need be done once to allow for the computation of $\Psi(x, t)$ for all $t \leq y$. We found this algorithm to be very accurate in practice, and it gave estimates for $\Psi(x, y)$ that are accureate to within a few percentage points of the correct value, well beyond the range guaranteed by Ennola's theorem.

At a high level, Algorithm HT [2] uses Newton's method to find the zero, α, of a continuous function. This value of α is then plugged into a formula for $\Psi(x, y)$. Our idea to improve the algorithm is to first find an approximation to α, called α_f, using the version of Algorithm HT that assumes the Riemann Hypothesis to

bound the error when estimating the distribution of primes, allowing for faster summing of functions of primes [3]. Then, starting from α_f, Newton's method is applied in the context of the original Algorithm HT, allowing for much faster convergence, usually requiring only one iteration in practice, yet providing the same level of accuracy as Algorithm HT. We present computational results showing this approach is effective in practice.

For our empirical study, we implemented five chosen algorithms to estimate $\Psi(x, y)$, listed below in order of optimality based on how large y is relative to x, starting with smaller y:

1. Buchstab's identity, which leads to a recursive algorithm that is very slow but gives exact values of $\Psi(x, y)$.
2. Our new algorithm based on Ennola's second theorem, as mentioned above.
3. Algorithm HT as mentioned above; either the old or new version.
4. Algorithm HT-fast, as described in [3].
5. A version of the Dickman ρ estimate, where

$$\Psi(x, y) \sim x\rho(u) + (1 - \gamma)\frac{x}{\log x}\rho(u - 1),$$

which is extremely fast but not as accurate as the other methods listed here. See [4, 2, 5].

We explored the ranges of applicability of each of these algorithms, juding both accuracy and performance in practice. We present recommendations for which algorithm to use, given values of x, y for $y \le x \le 2^{2500}$.

References

[1] V. Ennola. On numbers with small prime divisors. *Ann. Acad. Sci. Fenn. Ser. A I*, 440, 1969. 16pp.

[2] Simon Hunter and Jonathan P. Sorenson. Approximating the number of integers free of large prime factors. *Mathematics of Computation*, 66(220):1729–1741, 1997.

[3] Jonathan P. Sorenson. A fast algorithm for approximately counting smooth numbers. In W. Bosma, editor, *Proceedings of the Fourth International Algorithmic Number Theory Symposium (ANTS IV)*, pages 539–549, Leiden, The Netherlands, 2000. LNCS 1838.

[4] Gérald. Tenenbaum. *Introduction to Analytic and Probabilistic Number Theory*, volume 46 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, english edition, 1995.

[5] J. van de Lune and E. Wattel. On the numerical solution of a differential-difference equation arising in analytic number theory. *Mathematics of Computation*, 23:417–421, 1969.