An Algorithm to Find Sums of Consecutive Powers of Primes

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July 18, 2022

ANTS Poster Abstract

Let $\mathcal{S}_k(x)$ denote the set of integers $n \le x$ that can be written as a sum of the kth powers of consecutive primes. For example, $5^3 + 7^3 + 11^3 = 1799$ is an element of $\mathcal{S}_3(2000)$. Let $S_k(x)$ be $\#\mathcal{S}_k(x)$.

In this poster,

- We describe an algorithm that, given k and x, produces the elements of $\mathcal{S}_k(x)$ along with their representation. Its running time is linear in such representations; in practice this is linear in $S_k(x)$. The algorithm uses $O(kx^{1/k})$ space.

- We show that

$$S_k(x) \ll c_k \frac{x^{2/(k+1)}}{(\log x)^{2k/(k+1)}},$$

  where $c_k = (k^2/(k-1)) \cdot (k+1)^{1-1/k}$. This is a generalization of a bound for $S_2(x)$ proven by [4]. Their bound is explicit and ours is not. This is

also an upper bound on the number of arithmetic operations used by our algorithm.

- We apply our new algorithm to find exact values of $S_k(x)$ for various x and k, and give some examples of integers that can be written as sums of consecutive powers of primes in more than one way.

For example, we found 40 values of $n \le x = 10^{12}$ that have multiple representations as sums of consecutive squares of primes. The smallest such number is 14720439, which can be written as

$$941^2 + 947^2 + 953^2 + \cdots + 1031^2 + 1033^2$$

and as

$$131^2 + 137^2 + 139^2 + \cdots + 643^2 + 647^2.$$

We found exactly one example with differing powers:

$$\begin{aligned} 23939 &= 23^2 + 29^2 + 31^2 + 37^2 + 41^2 + 43^2 + 47^2 + 53^2 + 59^2 + 61^2 + 67^2 \\ &= 17^3 + 19^3 + 23^3. \end{aligned}$$

We found no integers that can be written as the sum of consecutive powers of primes in more than one way for any power larger than 2. We searched for cubes up to 10^{18}, fifth powers up to 10^{27}, and tenth and twentieth powers up to 10^{38}.

Our computations imply that the bound on $S_k(x)$ above is, in practice, accurate to within a constant factor near 0.6.

Note that $\mathcal{S}_2(5000)$ was computed by [4]; see also sequence A340771 at the On-Line Encyclopedia of Integer Sequences (OEIS.org) [1].

References

[1] The On-Line Encyclopedia of Integer Sequences. `https://oeis.org`.

[2] Eric Bach and Jeffrey O. Shallit. *Algorithmic Number Theory*, volume 1. MIT Press, 1996.

[3] Cathal O'Sullivan, Jonathan P. Sorenson, and Aryn Stahl. An algorithm to find sums of consecutive powers of primes, 2022. Available on arxiv.org at `https://arxiv.org/abs/2204.10930`.

[4] Janyarak Tongsomporn, Saeree Wananiyakul, and Jörn Steuding. Sums of consecutive prime squares. *Integers*, 22, 2022. A9.