TWISTS OF THE BURKHARDT QUARTIC THREEFOLD

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Abstract. We study twists of the Burkhardt quartic threefold over non-algebraically closed base fields of characteristic different from $2, 3, 5$. We show they all admit quartic models in projective four-space. We identify a Galois-cohomological obstruction that measures if a given twist is birational to a moduli space. This obstruction has implications for the rational points on these varieties. As a result, we see that all possible 3-level structures can be realized by abelian surfaces, whereas Kummer 3-level structures that group theoretically may be admissible, may not be realizable over certain base fields. We give an example of a Burkhardt quartic over a bivariate function field whose desingularization has no rational points at all.

Our methods are based on the representation theory of $\mathrm{Sp}_4(\mathbb{F}_3)$, Galois cohomology, and the classical algebraic geometry of the Burkhardt quartic.

1. Introduction and results

The Burkhardt quartic threefold

$$B^{(1)}\colon y_0(y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3) + 3y_1y_2y_3y_4 = 0,$$

has received significant study both classically over $\mathbb{C}$ (see [Bur91, Mas89, Cob17, Hun96]) and more recently arithmetically. For instance, in [BN18, CC20] the rationality and non-rationality of certain twists of the Burkhardt quartic over $\mathbb{Q}$ is established, while in [CCR20, BCGP21] it is remarked that twists of the Burkhardt quartic that parametrize abelian surfaces are unirational.

In this paper, we consider twists of $B^{(1)}$ over base fields of characteristic distinct from $2, 3, 5$. By a *twist of* $B^{(1)}$ we mean a variety B over k that, when base changed to a separable closure k^{sep}, is isomorphic to $B^{(1)}$. We refer to such a variety B as *a* Burkhardt quartic over k. This terminology suggests that B can indeed be realized as a quartic threefold in $\mathbb{P}^4$. This is true, but requires proof. It, and some other basic facts, follows quite directly from the representation theory of $\mathrm{Sp}_4(\mathbb{F}_3)$. We collect these in the following theorem. See Section 3.1 for the proof.

Theorem 1.1. *Let k be a field of characteristic distinct from $,2, 3, 5$ and let B be a twist of $B^{(1)}$.*

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(a) *B admits a quartic model in $\mathbb{P}^4$ with 45 singularities.*
(b) *B comes equipped with a rational map $\pi\colon M \to B$ of generic degree 6, where M is a Brauer-Severi variety of dimension 3. We write $\mathrm{Ob}(B)$ for the class of M in $\mathrm{Br}(k)$.*
(c) *$\mathrm{Ob}(B) \in \mathrm{Br}(k)$ is of period dividing 2 and of index dividing 4.*

It is well-known that $B^{(1)}$ is a birational model of the moduli space of principally polarized abelian surfaces with full 3-level structure. Outside the Hessian locus $\mathrm{He}(B)$ on B, a point α corresponds to the Jacobian $A_\alpha = \mathrm{Jac}(C_\alpha)$ of a genus 2 curve C_α, together with an isomorphism $\Sigma^{(1)} \to A_\alpha[3]$. Here, $\Sigma^{(1)} = (\mathbb{Z}/3\mathbb{Z})^2 \times (\mu_3)^2$ is equipped with a natural pairing $\Sigma^{(1)} \times \Sigma^{(1)} \to \mu_3$ and the isomorphism is compatible with the Weil-pairing on $A_\alpha[3]$.

The intersection $B \cap \mathrm{He}(B)$ consists of a union of 40 planes over k^{sep}, called j-planes.

The rational map $\pi\colon M \to B$, which is regular outside of $\mathrm{He}(B)$, has a moduli interpretation as well. It corresponds to marking an odd theta characteristic: a rational weierstrass point on C_α. The M here is referred to as the *Maschke* $\mathbb{P}^3$.

For our purposes, it is more natural to think of $B^{(1)}$ as a moduli space of Kummer surfaces $K_\alpha = A_\alpha/\langle -1\rangle$. These come with a marked singularity (the image of the identity element of A_α) as well as a Kummer 3-level structure $\overline{\Sigma}^{(1)} = \Sigma^{(1)}/\langle -1\rangle$.

In general a Kummer surface K over k, with one of the 16 singular points marked, is a quotient of an abelian surface A over k^{sep}. It does not fully determine A over k: if A admits a model over k, then any quadratic twist of A has a Kummer surface isomorphic to K as well. In fact, there may be no such abelian surface over k at all. This is measured by $\mathrm{Ob}(K) \in \mathrm{Br}(k)$ and is represented by a conic Q_K. Equivalently, this obstruction arises from the moduli determining the curve C_α. These moduli determine a curve of genus 0 with a degree 6 locus marked, but only if that genus 0 curve is actually a $\mathbb{P}^1$ can one realize a double cover ramified over the marked locus.

The Kummer 3-level structure already detects $\mathrm{Ob}(K)$. We establish that any Burkhardt quartic B parametrizes Kummer surfaces with prescribed Kummer 3-level structure, so it follows that the obstruction map is constant and hence is a function of B itself. We collect results about it, and implications for the rational points on B, in the theorem below, that we prove in Section 4.1.

Theorem 1.2. *Let B be a Burkhardt quartic over a field k of characteristic not 2, 3, 5.*
(a) *Then B is naturally birational to the moduli space of Kummer surfaces with a Kummer 3-level structure $\overline{\Sigma}$.*
(b) *If $\alpha \in B(k) \setminus \mathrm{He}(B)(k)$, then $\mathrm{Ob}(B) = \mathrm{Ob}(K_\alpha)$.*
(c) *If $\mathrm{He}(B) \cap B$ contains a j-plane defined over k then $\mathrm{Ob}(B) = 1$.*
(d) *If the Kummer 3-level structure $\overline{\Sigma}$ is a quotient of a full 3-level structure Σ over k, then $B(k)$ is Zariski-dense in B, and one can find a hyperelliptic curve with a rational Weierstrass point*

$$C\colon y^2 = x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$$

such that $\mathrm{Jac}(C)[3] \simeq \Sigma$.
(e) *If $\mathrm{Ob}(B)$ has index 4 then $B(k)$ consists of singular points and the desingularization of B has no k-rational points at all.*

We also show that $\mathrm{Ob}(B)$ can indeed be of index 1, 2, or 4; see Section 4.2.

Example 1.3.

(a) The standard model $B^{(1)}$ over $\mathbb{Q}$ has $\mathrm{Ob}(B^{(1)}) = 1$, which is of index 1.

(b) The symmetric model

$$B' \colon \sigma_1 = \sigma_4 = 0,$$

where σ_i is the standard degree i elementary symmetric function in six variables, has $\mathrm{Ob}(B') = (-3,-1)$, which over $\mathbb{Q}$ is of index 2.

(c) The model

$$\begin{aligned} B'' \colon & z_0^4 + 4z_0z_1^3 + 3z_1^4 + 3s^2z_2^4 + 3t^2z_3^4 + 3s^2t^2z_4^4 \\ & + 12sz_0z_1z_2^2 + 12tz_0z_1z_3^2 + 12stz_0z_1z_4^2 + 24stz_0z_2z_3z_4 + 24stz_1z_2z_3z_4 \\ & - 6sz_1^2z_2^2 - 6tz_1^2z_3^2 - 6stz_2^2z_3^2 - 6stz_1^2z_4^2 - 6s^2tz_2^2z_4^2 - 6st^2z_3^2z_4^2 = 0. \end{aligned}$$

has $\mathrm{Ob}(B'') = (-1,s) \otimes (s,t)$, which over $k = \mathbb{R}(s,t)$ has index 4. We have $B''(\mathbb{R}(s,t)) = \{(1 : -1 : 0 : 0 : 0)\}$, which is a singular point on B''. The blow-up of B'' at that point has no k-rational points at all.

We see that having $\mathrm{Ob}(B)$ of index 4 puts severe restrictions on the rational points on B. As Example 1.3(c) shows, there are Kummer 3-level structures that *a-priori* are admissible in the sense that they correspond to an element of $\mathrm{H}^1(k, \mathrm{PSp}_3(\mathbb{F}_3))$, but do not occur for a Kummer surface over k. This is in stark contrast to what happens with 3-level structures for abelian surfaces, where Theorem 1.1(a) guarantees that the corresponding moduli space is in fact unirational.

Over a number field, however, and in particular over $\mathbb{Q}$, index and period of Brauer group elements agree, so $\mathrm{Ob}(B)$ is of index at most 2 and we don't get a particular obstruction to $\mathbb{Q}$-rational points on B.

For instance for Example 1.3(b) it is not hard to find many rational points on B', including ones that do not lie in $\mathrm{He}(B')$. We establish in Section 4.3 the following.

Proposition 1.4. *The Burkhardt quartic defined by $B' \colon \sigma_1 = \sigma_4 = 0$ over $\mathbb{Q}$ is birational to the elliptic threefold in $\mathbb{P}^2 \times \mathbb{A}^2$, defined by*

$$\begin{aligned} C_{(u,v)} \colon & (u+v-1)XY(X+Y) + (-uv+u+v)(X^2+Y^2)Z \\ & + (-u^2 - 3uv + 3u - v^2 + 3v - 1)XYZ + (u^2v + uv^2 - uv)Z^3 \\ & + (u^2v - u^2 + uv^2 - 3uv + u - v^2 + v)(X+Y)Z^2 = 0, \end{aligned}$$

where $C_{(u,v)}$ has rational flex points $(1:0:0), (0:1:0), (1:-1:0)$. The map to B' is given by

$$(x1 : x_2 : x_3 : x_4 : x_5 : x_6) = (X : Y : -uZ : -vZ : Z : -X - Y + (u+v-1)Z).$$

We have that $C_{3/5,4}$ has a rational point $(20 : 2 : 15)$, and in fact has infinitely many rational points. Furthermore, $B'(\mathbb{Q})$ is Zariski-dense in B'.

With some modest experimentation we have not been able to find a twist of B over $\mathbb{Q}$ that did not have any rational points. This gives some mild circumstantial evidence for a possibly negative answer to the following question.

Question 1.5. Does there exist a Burkhardt quartic B for which $\mathrm{Ob}(B)$ has index 2 and for which the rational points do *not* lie Zariski-dense?

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2. Background

2.1. **Brauer groups and Brauer-Severi varieties.** In what follows, we frequently refer to the Brauer group $\mathrm{Br}(k)$ of a field k. There are many descriptions. It can be described as the Galois cohomology group $\mathrm{Br}(k) = \mathrm{H}^2(k, k^{\mathrm{sep}*})$. Elements of $\mathrm{Br}(k)$ also correspond to k-isomorphism classes of *Brauer-Severi* varieties: varieties that, over k^{sep}, are isomorphic to $\mathbb{P}^n$ for some $n \geq 0$. We refer to [GS06] for details; here we just review some standard terminology and result that we need in the rest of the text.

The *period* of an element in $\mathrm{Br}(k)$ is its order under the group structure of $\mathrm{Br}(k)$. We will only be dealing with elements of order dividing 2, i.e., we elements that lie in $\mathrm{Br}(k)[2] = \mathrm{H}^2(k, \mu_2)$, where μ_p stands for the p-th roots of unity.

The *index* of an element $\xi \in \mathrm{Br}(k)$ is the smallest n such that there is an extension L of degree n such ξ lies in the kernel of the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(L)$. Since a Brauer-Severi variety V is isomorphic to $\mathbb{P}^n$ if and only if it has a rational point L it is the smallest degree d such that there is a degree d extension L for which V has an L-rational point. The period always divides the index. However, for fields of higher cohomological dimension, such as $\mathbb{R}(s,t)$, the period can be strictly smaller than the index.

Elements of $\mathrm{Br}(k)$ also correspond to Brauer-equivalence classes of central simple algebras. The group law on $\mathrm{Br}(k)$ is induced by the tensor product on algebras. A famous theorem by Merkurjev-Suslin states that $\mathrm{Br}(k)[2]$ is generated by quaternion algebras. For $a, b \in k^*$ we write (a, b) for the quaternion algebra

$$(a, b) = k \oplus ik \oplus jk \oplus ijk, \text{ with } i^2 = a,\ j^2 = b,\ ij = -ji.$$

We also use (a, b) to denote its Brauer class in $\mathrm{Br}(k)$. The Brauer-Severi variety belonging to (a, b) is the conic $Q\colon z^2 - ax^2 - by^2 = 0$. Elements of $\mathrm{Br}(k)[2]$ of index at most 2 are exactly the ones that can be represented by a single quaternion algebras, i.e., the isomorphism classes of plane conics.

2.2. **Obstructions for genus** 2 **curves.** As is well-known, genus 2 curves and, equivalently, abelian surfaces, can have different fields of moduli and fields of definition: for a non-algebraically closed field k, one may have an isomorphism class of genus 2 curves over k^{sep} that is stable under $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, but does not contain any curves defined over k.

This phenomenon can be made very explicit, see [Mes91]. A genus 2 curve is geometrically determined by a degree 6 separated locus on a genus 0 curve. This data can be specified over k by a plane conic Q and cubic curve C (in fact, a 2-dimensional linear system of cubics) over k. The data only correspond to a genus 2 curve defined over k if Q is isomorphic to $\mathbb{P}^1$ over k, in which case an appropriate genus 2 curve is obtained as a double cover of $\mathbb{P}^1$, ramified over the degree 6 locus.

The isomorphism class of a conic Q over k, a Brauer-Severi variety of dimension 1, is an element of the Brauer group of k of index and period dividing 2. Writing $\mathcal{M}_2$ for the (coarse) moduli space of curves of genus 2, the construction above gives rise to a map

$$\mathrm{Ob}\colon \mathcal{M}_2(k) \to \mathrm{Br}(k)[2],$$

where a point $\alpha \in \mathcal{M}_2(k)$ can be represented by a genus 2 curve defined over k if and only if $\mathrm{Ob}(\alpha)$ vanishes.

2.3. **Principally polarized abelian surfaces and their Kummer surfaces.** Let X be a curve of genus 2 over a field k. Then $\mathrm{Jac}(X)$ is a principally polarized abelian surface over k, and by the Torelli theorem, X can be recovered from $\mathrm{Jac}(X)$: There is an injective morphism between moduli spaces $\mathcal{M}_2 \to \mathcal{A}_2$.

Associated to a principally polarized abelian surface A is its *Kummer surface* $\mathrm{Kum}(A) = A/\langle -1\rangle$, obtained by identifying points with their inverses. The fixed locus of the inversion map, the 2-torsion $A[2]$, is 0-dimensional of degree 16 and maps onto the singular locus of $\mathrm{Kum}(A)$. In addition, on of the singular points on $\mathrm{Kum}(A)$ is distinguished: it is the image of the identity element of A.

If $A = \mathrm{Jac}(X)$, then $\mathrm{Kum}(A)$ admits a quartic model in $\mathbb{P}^3$ (a Kummer quartic), and conversely, any quartic surface K in $\mathbb{P}^3$ with 16 nodal singularities and a distinguished node can be recognized as $\mathrm{Kum}(A)$ for some A over k^{sep}. An explicit, classic construction to do so goes as follows.

The projective dual K^* of K is again a Kummer quartic surface. The 16 singularities of K^* correspond to 16 *tropes* on K: planes that intersect K in a double-counting conic. Each trope passes through 6 nodes and each node lies on 6 tropes, forming the classical (16_6) *Kummer configuration.*

Note that K has a distinguished node and, by duality, K^* has a distinguished trope. It is a classic result that over an algebraically closed base field k, the surface K is isomorphic K^*, but this shows that this need not hold if k is not algebraically closed.

The trope on K^* cuts out a plane conic with six marked points: the nodes the trope passes through. Equivalently, we consider the tangent cone to K at the distinguished node. The six tropes passing through it intersect the tangent cone in lines through the node. Projection from the node yields a plane conic Q_K with six marked points.

If $Q_K \simeq \mathbb{P}^1$ then this data determines a genus 2 curve X (up to quadratic twist) such that $K = \mathrm{Kum}(\mathrm{Jac}(X))$. For a Kummer surface K over k, we write $\mathrm{Ob}(K)$ for the isomorphism class of Q_K in $\mathrm{Br}(k)$. Indeed, every point $\alpha \in \mathcal{M}_2(k)$ corresponds to a Kummer quartic surface K_α such that $\mathrm{Ob}(\alpha) = \mathrm{Ob}(K_\alpha)$ and such that if $\mathrm{Ob}(\alpha) = 0$, then $\mathrm{Kum}(\mathrm{Jac}(X_\alpha)) = K_\alpha$. This can be checked by considering a quadratic extension L of k such that $\mathrm{Res}_L(\mathrm{Ob}(\alpha)) = 0$, construct X_α over L, and confirming that K_α can be descended to k.

Proposition 2.1. *Let K be a sufficiently general quartic Kummer surface with distinguished node. Then K has trivial automorphism group preserving the node.*

Proof. As described above, we can recover, at least over an extension where $\mathrm{Ob}(K)$ is trivial, a cover $A \to K$ from K with its node. But then an automorphism $K \to K$ lifts to at least a birational map $A \to A$. This extends to an automorphism. A sufficiently general A (in fact, an open part of $\mathcal{A}_2$) has automorphism group $\langle \pm 1\rangle$, but that means the induced automorphism on K is just the identity. □

Remark 2.2. As is shown in [CQ05], for a point $\alpha \in \mathcal{A}_2(k)$ representing an abelian variety A_α over k^{sep} with an automorphism group larger than just μ_2, the variety A_α can be descended to a model over k. In that sense, $\mathrm{Ob}(\alpha) = 0$. However, in the case where A_α is a product of elliptic curves, there are extra automorphisms from negation on one of the factors. As a result, K_α inherits a non-trivial automorphism and the isomorphism class of K_α over k is not uniquely determined by α. Indeed, there there can be an obstruction $\mathrm{Ob}(K)$ in those cases that does not factor through $\mathrm{Ob}(\alpha)$.

We can see this in the following way. Suppose that A is the Weil restriction of an elliptic curve E over a quadratic extension $L = k[\sqrt{r}]$, possibly split. The quotient $E \to \mathbb{P}^1$ by -1 induces a degree 4 map from A to the Weil restriction V of $\mathbb{P}^1$. If we write

$$E^{(\delta)} \colon \delta y^2 = f(x) = x^3 + a_2x^2 + a_4x + a_6,$$

where $\delta \in L^\times$ prescribes a quadratic twist of E over L, then we can model V as a quadric in $\mathbb{P}^3$ with affine model $x_0^2 - rx_1^2 - z = 0$, related to $E^{(\delta)}$ by $x = x_0 + x_1\sqrt{r}$. We can get a degree 16 model $K^{(d)}$ for $\mathrm{Kum}(A^{(\delta)})$ in weighted 9-dimensional projective space with coordinates

$$(1 : x_0 : x_1 : x_0^2 : x_0x_1 : x_1^2 : zx_0 : zx_1 : z^2 : w),$$

with weights $1, \ldots, 1, 2$, which is a double cover of the degree 2 Segre embedding of V, with additional equation

$$dw^2 = N_{L/k}(f(x_0 + x_1\sqrt{r})),$$

where $d = N_{L/k}(\delta)$. The right hand side is indeed quartic in x_0, x_1, z thanks to the defining relation for V. We see that the isomorphism class of $K^{(d)}$ only depends on the class of d in $k^\times/k^{\times 2}$. On the other hand, we see that $K^{(d)}$ only admits a cover by $A^{(\delta)}$ if there is a non-zero solution to the norm equation $u_0^2 - ru_1^2 = du_2^2$, i.e., if a conic is isomorphic to $\mathbb{P}^1$.

2.4. **3-Level structure.** The 3-torsion on a principally polarized abelian surface A is a 0-dimensional group scheme $A[3]$ of degree 3^4, together with a perfect alternating pairing $A[3] \times A[3] \to \mu_3$.

Let Σ be such a group scheme, equipped with pairing. A 3-level structure on a principally polarized abelian surface A is an isomorphism $\Sigma \to A[3]$ compatible with the pairings on either side. One such group scheme is $\Sigma^{(1)} = (\mathbb{Z}/3\mathbb{Z})^2 \times (\mu_3)^2$, with the pairing induced by the fact that $(\mu_3)^2$ is the Cartier dual to $(\mathbb{Z}/3\mathbb{Z})^2$.

The automorphism group of $\Sigma^{(1)}$ is isomorphic to $\mathrm{Sp}_4(\mathbb{F}_3)$. The twisting principle (see [Mil80, III.4]) implies that the isomorphism class of Σ over k, being a twist of $\Sigma^{(1)}$, is classified by the Galois cohomology set $\mathrm{H}^1(k, \mathrm{Sp}_4(\mathbb{F}_3))$.

We write $\overline{\Sigma}$ for $(\Sigma - \{0\})/\langle \pm 1 \rangle$. It is a degree 40 scheme, together with a pairing $\overline{\Sigma} \times \overline{\Sigma} \to \{0, 1\}$, determined by whether the corresponding representatives in Σ pair trivially. The automorphism group $\mathrm{Sp}_4(\mathbb{F}_3)$ acts on $\overline{\Sigma}$, with the center (generated by -1) acting trivially. Hence the action factors through $\mathrm{PSp}_4(\mathbb{F}_3)$.

Remark 2.3. The pairing information on $\overline{\Sigma}$ can be almost captured by 40 subsets of cardinality 4, from the 40 maximal isotropic subspaces of Σ. The group of permutations preserving this incidence structure is $\mathrm{PGSp}_4(\mathbb{F}_3)$, which contains $\mathrm{PSp}_4(\mathbb{F}_3)$ as an index 2 subgroup.

Let A be an abelian surface with a 3-level structure $\Sigma \to A$. Since multiplication-by-3 commutes with negation, it induces a well-defined map $K \to K$, which we call *pseudo-multiplication by* 3. The 3-level structure on A induces an isomorphism between $\overline{\Sigma} \cup 0$ and the fibre of the distinguished point. We call this a Kummer 3-level structure on K. A Kummer surface has pseudo-multiplication maps, regardless of whether $\mathrm{Ob}(K)$ is trivial. Hence we have a Kummer 3-level structure on it as well.

The possible Kummer 3-level structures are classified by $\mathrm{H}^1(k, \mathrm{PSp}_4(\mathbb{F}_3))$. Taking cohomology of the short exact sequence

$$1 \to \mu_2 \to \mathrm{Sp}_4(\mathbb{F}_3) \to \mathrm{PSp}(\mathbb{F}_3) \to 1$$

gives a map

$$\mathrm{H}^1(k, \mathrm{Sp}_4(\mathbb{F}_3)) \to \mathrm{H}^1(k, \mathrm{PSp}_4(\mathbb{F}_3)) \xrightarrow{\mathrm{Ob}} \mathrm{H}^2(k, \mu_2)$$

For a Kummer 3-level structure $\overline{\Sigma}$ we write $\mathrm{Ob}(\overline{\Sigma})$ for its class in $\mathrm{H}^2(k, \mu_2)$.

By Proposition 2.1, we see that for a point $\alpha \in \mathcal{A}_2(k)$ for which K_α is sufficiently general, we have that K_α and $\overline{\Sigma}$ are determined by α and hence that $\mathrm{Ob}(K_\alpha) = \mathrm{Ob}(\overline{\Sigma})$.

As explained in Remark 2.2, for α that correspond to products of elliptic curves, the isomorphism class of the Kummer surface is only determined up to twist, but the Kummer 3-level structure determines the twist $K_{\alpha,\overline{\Sigma}}$. In this case, it can be checked that once again, $\mathrm{Ob}(K_{\alpha,\overline{\Sigma}}) = \mathrm{Ob}(\overline{\Sigma})$.

2.5. **Moduli spaces with 3-level structure.** We write $\mathcal{A}_2(\Sigma)$ for the moduli space of principally polarized abelian surfaces A with 3-level structure $\Sigma \to A$. Since the (-1)-automorphism on Σ is the restriction of -1 on A, the isomorphism class of $\mathcal{A}_2(\Sigma)$ only depends on $\overline{\Sigma}$, and therefore we write $\mathcal{A}_2(\overline{\Sigma})$.

Indeed, from

$$\mathrm{H}^1(k, \mu_2) \to \mathrm{H}^1(k, \mathrm{Sp}_4(\mathbb{F}_3)) \to \mathrm{H}^1(k, \mathrm{PSp}_4(\mathbb{F}_3))$$

we see that different Σ, Σ' map to isomorphic $\overline{\Sigma}$ exactly when they are quadratic twists, and taking quadratic twists of an Abelian variety A will correspondingly twist its 3-torsion.

In particular, we see that the level-structure-forgetting morphism $\mathcal{A}_2(\overline{\Sigma}) \to \mathcal{A}_2$ is Galois with automorphism group $\mathrm{PSp}_4(\mathbb{F}_3)$. Hence, for *any* class $\overline{\Sigma}$ in $\mathrm{H}^1(k, \mathrm{PSp}_4(\mathbb{F}_3))$ we have a corresponding twist $\mathcal{A}_2(\overline{\Sigma})$, which we can then consider as a moduli space of Kummer surfaces with 3-level structure. The space comes with an obstruction map

$$\mathrm{Ob}\colon B(k) \to \mathrm{Br}(k);\ \alpha \mapsto \mathrm{Ob}(K_\alpha),$$

which is constant $\mathrm{Ob}(\overline{\Sigma})$.

3. Representation-theoretic description of the Burkhardt quartic threefold

For now, we fix the 3-level structure $\Sigma = \Sigma^{(1)} = (\mathbb{Z}/3\mathbb{Z})^2 \times (\mu_3)^2$ and the form Γ of $\mathrm{Sp}_4(\mathbb{F}_3)$ over k that is its automorphism group. The group affords two faithful irreducible 4-dimensional representations: ρ_4 and its dual $\rho_4^\vee$ the complex conjugate.

χ_i	$\chi_i(A_1)$	$\chi_i(A_2)$	$\chi_i(A_3)$
ρ_4	-4	$2\zeta+1$	$3\zeta+2$
$\rho_4^\vee$	-4	$-2\zeta-1$	$-3\zeta-1$
ρ_5	5	0	$-3\zeta-1$
$\rho_5^\vee$	5	0	$3\zeta+2$
ρ_{10}	10	-1	$-3\zeta-5$
$\rho_{10}^\vee$	10	-1	$3\zeta-2$
χ_7	-20	0	7
ρ_{20}	20	1	2
χ_9	-20	0	$3\zeta-5$
χ_{10}	-20	0	$-3\zeta-8$
χ_{11}	-20	$2\zeta+1$	$6\zeta+1$
χ_{12}	-20	$-2\zeta-1$	$-6\zeta-5$
χ_{13}	30	-1	3
ρ_{30}	30	0	$-9\zeta-6$
$\rho_{30}^\vee$	30	0	$9\zeta+3$

TABLE 1. Characters of $\mathrm{Sp}_4(\mathbb{F}_3)$ up to 30

Explicit generators for ρ_4 are

$$A_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$A_2 = \frac{1}{3}\begin{pmatrix} 3\zeta & 0 & 0 & 0 \\ 0 & \zeta+2 & \zeta+2 & -\zeta+1 \\ 0 & \zeta+2 & \zeta-1 & -\zeta-2 \\ 0 & -\zeta+1 & -\zeta-2 & \zeta+2 \end{pmatrix},$$

$$A_3 = \frac{1}{3}\begin{pmatrix} 2\zeta+1 & -2\zeta-1 & 0 & \zeta+2 \\ \zeta-1 & 2\zeta+1 & 0 & -\zeta+1 \\ 0 & 0 & 3\zeta+3 & 0 \\ \zeta+2 & 2\zeta+1 & 0 & 2\zeta+1 \end{pmatrix}.$$

It induces a projective representation of $\overline{\Gamma} = \Gamma/\langle -1 \rangle$, which is a form of $\mathrm{PSp}_4(\mathbb{F}_3)$. It can be described as the automorphism group of a *Witting configuration* in $M = \mathbb{P}^3$, consisting of 40 points and 40 planes, such that each point lies in 12 planes and each plane passes through 12 points.

Table 1 lists the characters of $\mathrm{Sp}_4(\mathbb{F}_3)$ up to degree 30, together with the character values at the generators. Of particular note is that the characters of degree up to 10 are almost completely determined by their degree: there are at most two of each, in which case one is the complex conjugate of the other.

A simple character computation, or an explicit computation with the given generators A_1, A_2, A_3, shows that $S^2\rho_4 = \rho_{10}$, which is a representation of $\overline{\Gamma}$. This shows us that on the second Veronese embedding $v_2 \colon M \to \mathbb{P}^9$, the projective action of $\overline{\Gamma}$ is induced by a linear representation of $\overline{\Gamma}$ itself.

Further character computations show that $S^4\rho_4 = \rho_5 \oplus \rho_{30}$ and $S^2\rho_{10} = \rho_5 \oplus \rho_{30} \oplus \rho_{20}$. Indeed, the image of v_2 is described by a 20-dimensional space of quadrics on $\mathbb{P}^9$, corresponding to ρ_{20}. The representation ρ_5 gives rise to a rational map $\pi\colon M \dashrightarrow \mathbb{P}^4$, defined by a linear system of degree 4 and dimension 4 that vanishes on the Witting configuration in M. Alternatively, we can consider it as a linear system of quadratic forms on $\mathbb{P}^9$, via v_2. In fact, a further character computation shows that we can recover ρ_{10} from ρ_5 via $\bigwedge^2 \rho_5 = \rho_{10}^\vee$.

For $\Gamma = \langle A_1, A_2, A_3 \rangle$ as defined above, we can write $(t_1 : t_2 : t_3 : t_4)$ for the coordinates on M. The linear system corresponding to ρ_5 as a component of $S^4\rho_4$ is generated by

$$\begin{aligned}
Y_0 &= 3t_1t_2t_3t_4,\\
Y_1 &= t_1(t_2^3 + t_3^3 - t_4^3),\\
Y_2 &= -t_2(t_1^3 + t_3^3 + t_4^3),\\
Y_3 &= t_3(-t_1^3 + t_2^3 + t_4^3),\\
Y_4 &= t_4(t_1^3 + t_2^3 - t_3^3)
\end{aligned}$$

With this description of π, the image in $\mathbb{P}^4$ is dense in

$$B\colon y_0(y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3) + 3y_1y_2y_3y_4 = 0.$$

Hence, we recover the Burkhardt quartic threefold.

Intersection of B with its hessian $\mathrm{He}(B)$ yields a locus that over k^{sep} consists of 40 planes, called j-planes. The action of $\overline{\Gamma}$ on these is conjugate to the action of $\mathrm{PSp}_4(\mathbb{F}_3)$ on the cyclic subgroups of Σ of order 3. These are in bijection with $\overline{\Sigma}$. The pairing information is also reflected in the j-plane configuration: planes that pair trivially meet in a line and others meet in a point.

There is a synthetic description of the modular interpretation of B, see [Cob17, Hun96]. Let α be a point in $B \setminus \mathrm{He}(B)$. We write $P_\alpha^{(4-d)}$ for the degree d polar of B at α. Then $P_\alpha^{(3)} \cap P_\alpha^{(2)}$ is a cone over a plane conic Q_α, and the cubic $P_\alpha^{(1)}$ cuts out a degree six locus on Q_α. In fact, the enveloping cone at α of $P_\alpha^{(1)}$ yields a cone over a dual Kummer surface K_α^*, with $P_\alpha^{(3)}$ projecting to the distinguished trope. The j-planes project to tangent planes of K_α^* and hence yield points on its dual K_α, marking a Kummer 3-level structure $\overline{\Sigma}$ on K_α. It follows that $\mathrm{Ob}(\alpha) = \mathrm{Ob}(\overline{\Sigma})$.

The rational map $\pi\colon M \dashrightarrow B$ has generic degree 6 and also has a modular interpretation: outside $\mathrm{He}(B)$ it corresponds to the choice of an odd theta-characteristic, or, equivalently, a Weierstrass point on the genus 2 curve of which A_α is the Jacobian. In recognition of the work Maschke did on these spaces [Mas89], it is sometimes referred to as the *Maschke* $\mathbb{P}^3$.

As discussed before, for each $\xi \in \mathrm{H}^1(k, \overline{\Gamma})$ we get a different Kummer 3-level structure $\overline{\Sigma}^{(\xi)}$, and a corresponding form $\overline{\Gamma}^{(\xi)}$. Note that thanks to Hilbert 90, any representation $\rho\colon \overline{\Gamma} \to \mathrm{GL}_n$ gives rise to a corresponding representation $\rho^{(\xi)}\colon \overline{\Gamma}^{(\xi)} \to \mathrm{GL}_n$.

That means that $\overline{\Gamma}^{(\xi)}$ affords representations corresponding to $\rho_5, \rho_{10}, \rho_{20}$. In particular, we get a twist $B^{(\xi)} \subset \mathbb{P}^4$, together with a 3-dimensional Brauer-Severi variety

$M^{(\xi)} \subset \mathbb{P}^9$ and a degree 6 rational map $\pi^{(\xi)}\colon M^{(\xi)} \to B^{(\xi)}$. Note that $M^{(\xi)}$ is isomorphic to $\mathbb{P}^3$ precisely when the action of $\overline{\Gamma}^{(\xi)}$ can be lifted to a 4-dimensional linear representation, i.e., when ξ can be lifted to $\mathrm{H}^1(k, \Gamma)$. It follows that the isomorphism class of $M^{(\xi)} \subset \mathbb{P}^9$ as a Brauer-Severi variety is the image $\mathrm{Ob}(\overline{\Sigma}^{(\xi)}) = \mathrm{Ob}(\xi) \in \mathrm{H}^2(k, \mu_2)$.

3.1. **Proof of Theorem 1.1.**

Part (a): : As described above, the automorphism group $\mathrm{Aut}(B)$ of B is some form of $\mathrm{PSp}_4(\mathbb{F}_3)$). Thus, by Hilbert 90, there is a representation ρ_{10} of $\mathrm{Aut}(B)$, giving a linear projective action of $\mathrm{Aut}(B)$ on $\mathbb{P}^9$. Similarly, the decomposition $S^2\rho_{10} = \rho_5 \oplus \rho_{30} \oplus \rho_{20}$ yields a $\mathrm{Aut}(B)$-stable 19-dimensional linear system of quadrics on $\mathbb{P}^9$ defining a 3-dimensional Brauer-Severi variety $M \subset \mathbb{P}^3$, together with a covariant map $\pi\colon M \to \mathbb{P}^4$ from ρ_5. Its image then yields a quartic model for B. Over k^{sep}, this model differs from $B^{(1)}$ by a linear transformation, so B also has a singular locus of dimension 0 and degree 45.

Part (b): : We have already constructed the map π above. Base changing to k^{sep} does not change its generic degree, and there it agrees with the standard expression of the Maschke $\mathbb{P}^3$ over B.

Part (c): : By definition, $\mathrm{Ob}(B)$ is the class of the Brauer-Severi variety M. From the cohomological description, it is clear that $\mathrm{Ob}(B) \in H^2(k, \mu_2) = \mathrm{Br}(k)[2]$, so its period divides 2. Since M is a Brauer-Severi variety of dimension 3, its endomorphism ring is a 16-dimensional central simple algebra. But that means it is a tensor product of at most two quaternion algebras, so it can be split by the composite of splitting fields of each of the quaternion algebras, which gives extension of degree at most 4.

Remark 3.1. The moduli interpretation of B extends to products of elliptic curves as well: the blow-up of each of the 45 nodal singularities of B yields a component of the locus of $A_2(3)$ corresponding to products of elliptic curves. Indeed, $\mathrm{PSp}_3(\mathbb{F}_3)$ has a unique conjugacy class of index 45 subgroups, which are the stabilizers of decompositions of its standard representation into non-isotropic 2-dimensional subspaces.

The tangent cone of a node s on B is a cone over a non-singular quadric $V \subset \mathbb{P}^3$. Each node lies on eight j-planes, which map to four lines of each ruling on V. A choice of point α on V marks a distinguished point and one line from each ruling by intersection with the tangent plane. Each 4-tuple of lines cut out by j-planes cuts out the locus of a 3-division polynomial on one of the lines which, together with the marked intersection point, determines a k^{sep}-isomorphism class of an elliptic curve. As explained in Remark 2.2, this determines an elliptic Kummer surface up to twist, and $\overline{\Sigma}$ somehow encodes which one. We have not found a direct way of reading off the full information of the Kummer surface in this situation, but on general principles we know its obstruction will be $\mathrm{Ob}(\overline{\Sigma})$.

4. Period-index questions about obstructions

4.1. **Proof of Theorem 1.2.**

Part (a): Note that the isomorphism classes of Burkhardt quartics as well as of Kummer 3-level structures are classified by $\mathrm{H}^1(k, \mathrm{PSp}_4(\mathbb{F}_3))$. The synthetic description furthermore gives a way, given a point α on $B \backslash \mathrm{He}(B)$, to construct a Kummer surface K_α with the requisite level structure. In particular, we can do so at the generic point, to get a universal family over an open part of B.

Part (b): As we noted, for a Kummer 3-level structure $\overline{\Sigma}$ on a quartic Kummer surface, we have $\mathrm{Ob}(\overline{\Sigma}) = \mathrm{Ob}(K)$. Furthermore, we have $\mathrm{Ob}(B) = \mathrm{Ob}(\overline{\Sigma})$.

Part ((c)): By [BN18, Proposition 2.8], the choice of j-plane allows the construction of a cubic genus 1 curve together with a cubic map to $\mathbb{P}^1$, such that C_α is the discriminant curve of the cubic extension. This directly determines a model of C_α over k, so there is no obstruction for C_α and therefore $\mathrm{Ob}(B) = 1$.

Part (d): Note that $\mathrm{Ob}(\overline{\Sigma}) = \mathrm{Ob}(B)$ is the class of the Maschke M associated to B. Hence, if it is trivial then $M \simeq \mathbb{P}^3$, and $\pi\colon \mathbb{P}^3 \to B$ yields a unirational map. The image $\pi(\mathbb{P}^3(k))$ is then Zariski-dense. Points in the image correspond to Jacobians of genus 2 curves with a marked Weierstrass point, i.e., curves that admit a quintic affine model.

Part (e): If $\alpha \in B(k) \setminus \mathrm{He}(B)(k)$, then $\mathrm{Ob}(B) = \mathrm{Ob}(K_\alpha)$ is represented by a conic Q_α, and hence is of index at most 2. Hence, if $\mathrm{Ob}(B)$ is of index 4 then any rational point on B must lie in $\mathrm{He}(B)$. By (c) we know that for the field of definition L of any j-plane, the restriction of $\mathrm{Ob}(B)$ to L is trivial. If the index of $\mathrm{Ob}(B)$ is 4, then it follows that L has degree at least 4 and hence that any rational point on $\mathrm{He}(B)$ must lie on at least four j-planes, the conjugates. But the only points that lie on more than two j-planes are the singular points of B. Furthermore, by Remark 3.1 we see that the special fiber of the blow-up of B at any one of these singularities has a modular interpretation as well. By Remark 2.2 we see that any rational point on it would lead to a representative of $\mathrm{Ob}(B)$ of index at most 2, which would contradict that its index is 4.

4.2. **Proof of Example 1.3.** Part (a) follows because over finite fields $\mathrm{Br}(k) = 0$ and for infinite fields there are abelian varieties with 3-torsion structure $(\mathbb{Z}/3\mathbb{Z})^2 \times (\mu_3)^2$.

For Part (b) we let $\sigma_1, \sigma_4 \in k[x_1, \ldots, x_6]$ be the elementary symmetric functions of degrees 1 and 4 respectively. Then $B'\colon \sigma_1 = \sigma_4 = 0$ is also a Burkhardt quartic threefold, lying in the hyperplane $\sigma_1 = 0$ inside $\mathbb{P}^5$. We have $\alpha = (40 : -30 : -8 : -5 : 3 : 0) \in B'(\mathbb{Q})$ and Q_α is isomorphic to the plane conic $3x^2 + y^2 + z^2 = 0$. This conic is not isomorphic to $\mathbb{P}^1$ over $\mathbb{Q}$.

For Part (c) we take $k = \mathbb{R}(s,t)$, a bivariate function field. We take a twist of $B^{(1)}$ that is isomorphic to $B^{(1)}$ over $k(\sqrt{s}, \sqrt{t})$, by setting

$$\begin{aligned}
y_0 &= z_0 \\
y_1 &= z_1 + z_2\sqrt{s} + z_3\sqrt{t} + z_4\sqrt{st} \\
y_2 &= z_1 - z_2\sqrt{s} + z_3\sqrt{t} - z_4\sqrt{st} \\
y_3 &= z_1 + z_2\sqrt{s} - z_3\sqrt{t} - z_4\sqrt{st} \\
y_4 &= z_1 - z_2\sqrt{s} - z_3\sqrt{t} + z_4\sqrt{st}.
\end{aligned}$$

This yields the model B'' as stated.

Note that $\mathrm{Ob}(B'')$ is an element of period 2 and that it trivializes upon base change to $k(\sqrt{s}, \sqrt{t})$. Also note that B'' is actually defined over $k[s,t]$ and has good reduction outside $st = 0$. It follows that $\mathrm{Ob}(B) \in \mathrm{Br}(k[s, s^{-1}, t, t^{-1}])$. This group is generated by $(-1,-1), (-1,s), (-1,t), (s,t)$.

We note that the product of restriction maps

$$\mathrm{Br}(k) \to \mathrm{Br}(k(\sqrt{s})) \times \mathrm{Br}(k(\sqrt{t})) \times \mathrm{Br}(k(\sqrt{st}))$$

is injective on $\mathrm{Br}(k[s,s^{-1},t,t^{-1}])$. We compute the restriction to $\mathrm{Br}(k(\sqrt{s}))$ by specializing to $s=1$. The intersection with $z_3 = z_4 = 0$ yields a genus 0 curve on B'', with the point $\alpha = (16 : -31 : 9 : 0 : 0)$ outside $\mathrm{He}(B'') = 0$. We find that Q_α is equivalent to $tX^2 + Y^2 + Z^2 = 0$, and therefore that the restriction of $\mathrm{Ob}(B'')$ to $k(\sqrt{s})$ is $(-1,t)$. Symmetry implies that $\mathrm{Ob}(B'') = (-1,t) \otimes (-1,s) \otimes (s,t) = (-1,s) \otimes (-s,t)$. We describe two ways to show that this class is of index four.

First, one can simply enumerate all the classes of index at most two, since they will be of the form (a,b), where a,b lie in the multiplicative group generated by $\{-1,s,t\}$. Given that $(a,a) = (-1,a)$, we see there are $\binom{7}{2}+1$ choices, but many represent equivalent classes. The classes that are not covered (and hence must be of index four) are

$$(-1,-1)\otimes(s,t),\ (-1,-1)\otimes(s,-t),\ (-1,-1)\otimes(-s,t),\ (-1,s)\otimes(-s,t).$$

Alternatively, one use that a biquaternion algebra is of index four if and only if its *Albert form* is anisotropic, see [Lam73, Albert's Theorem 4.8]. In fact, Albert's original example [Alb32, Theorem 1] applies directly to $(s,s)\otimes(t,st)$, which is equivalent to our algebra.

If follows from Theorem 1.2(e) that all rational points of B'' lie in the singular locus. The only rational point there is $(1 : -1 : 0 : 0 : 0)$.

In order to show that the desingularization of B'' has no rational points at all, we can also directly look at the blow-up. The tangent cone to B'' at $(1 : -1 : 0 : 0 : 0)$ is the affine cone over the quadric

$$su_0^2 + tu_1^2 + stu_2^2 - 3u_3^2 = 0,$$

which is indeed easily checked to have no k-rational points.

4.3. **Proof of Proposition 1.4.** First note that B' has 15 rational singularities, constituting the orbit of $(1 : -1 : 0 : 0 : 0 : 0)$ under the action of S_6 on the coordinates. These singularities form 20 triples of collinear points. The lines lie in $\mathrm{He}(B') \cap B'$. For instance, the singularities $(1 : -1 : 0 : 0 : 0 : 0), (1 : 0 : 0 : 0 : 0 : -1), (0 : 1 : 0 : 0 : 0 : -1)$ lie on the line $L_{345} = x_3 = x_4 = x_5 = \sigma_1 = 0$. We consider the 2-dimensional linear system of planes $V_{u,v}$ in $\sigma_1 = 0$ containing this line, defined by

$$V_{u,v} \colon x_3 - ux_5 = x_4 - vx_5 = \sigma_1 = 0.$$

The intersection $V_{u,v} \cap B'$ decomposes into the line and the plane cubic $C_{u,v}$ stated in the proposition. It is straightforward to check that the singularities give rise to three collinear flexes on $C_{u,v}$. By choosing one of those flexes as zero-section, we see that $C_{u,v}$ is an elliptic threefold with 3-torsion. This yields a birational elliptic fibration on B'. In fact, the different choices of triples of collinear singularities on B' gives us 20 such fibrations.

In order to establish density of rational points on B' we use a standard trick combining these multiple fibrations.

It is straightforward to check that $V_{3/5,4}$ passes through the point $P_0 = (20 : 2 : -9 : -60 : 15 : 32) \in B'(\mathbb{Q})$ and that it yields a non-torsion point on $C_{3/5,4}$. Since the rational points on the line $x_3 = x_4 = x_5 = \sigma_1 = 0$ are definitely dense, this gives that rational points on B' are dense in the intersection of B' with the plane spanned by P_0 and L_{345}.

Next, we pick the fibration generated by planes through $L_{245}\colon x_2 = x_4 = x_4 = \sigma_1 = 0$. The fibers that intersect the plane above form a one-dimensional family of elliptic curves, of which a Zariski-dense set passes through an extra rational point. Only finitely many of these are torsion, so most of those fibers have infinitely many rational points themselves. This yields Zariski,density of rational points in B' intersected by the 3-space spanned by P_0, L_{345} and L_{245}.

We repeat this trick once more using the fibration generated by planes through $L_{145}\colon x_1 = x_4 = x_5 = \sigma_1$. The multi-section obtained by intersecting with the 3-space above is generically non-torsion, so there is a proper sublocus where it reduces to torsion of order, say, at most 12 (the largest that can occur over $\mathbb{Q}$). The result above shows there is a Zariski-dense set of fibers that have an extra rational point arising from this multi-section, and it follows a Zariski-dense subset has positive rank. This yields Zariski-density of rational points on B'.

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