Effective Chebotarev density theorems for families of number fields without GRH

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joint with Lillian Pierce and Caroline Turnage-Butterbaugh

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"Effective": error term and lower bound on X

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- want to apply in families, β_0 not uniform, need to remove effect of exceptional zero
- need to apply to smaller X, e.g. $X = D_K^{\epsilon}$, need bigger zero-free region

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- A_4 fields, all ramification order 3
- cyclic fields, all ramification total

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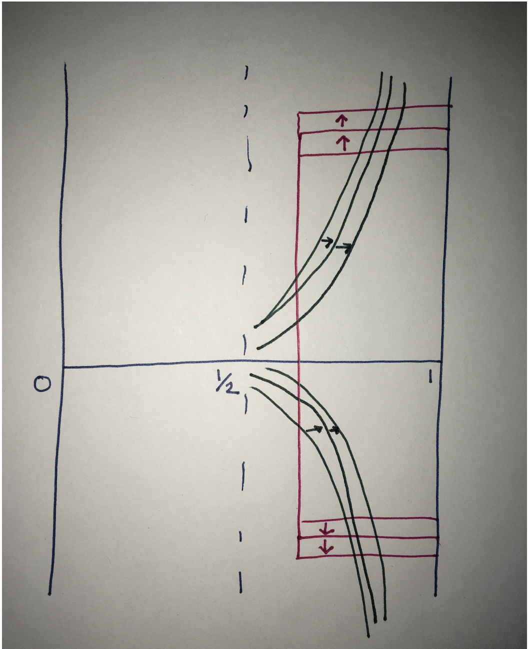
Zero-free regions



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- convexity and joint convexity bounds

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- Consider the family $L_{bad} L_i$ for cuspidal L_i

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Task: counting number fields with fixed subfields (corresponding to $\ker \rho$)

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- main tool: pointwise upper bounds on $\#\{K : |D_K| = X\}$ and control ramification

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Upper bounds with fixed discriminant, for dihedral D_p:

- cyclic p extensions of quadratic fields

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