THORSTEN KLEINJUNG*

BENJAMIN WESOLOWSKI*

A NEW PERSPECTIVE ON THE POWERS OF TWO DESCENT
for discrete logarithms in finite fields

PRESENTED AT ANTS-XIII, MADISON, WI, USA, ON THE 20/07/2018 BY BENJAMIN WESOLOWSKI

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A ZIGZAG DESCENT

HEURISTIC AND RIGOROUS ALGORITHMS FOR DLP

Discrete logarithm problem (DLP) in finite fields of fixed characteristic ($\mathbb{F}_{p^n}$ with p fixed and $n \to \infty$... think $\mathbb{F}_{2^n}$):

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they need to be understood better

QUASI-POLYNOMIAL ALGORITHMS FOR DLP

- First heuristic quasi-poly. algorithm discovered by Barbulescu, Gaudry, Joux, Thomé [BGJT14]

[BGJT14] R. Barbulescu, P. Gaudry, A. Joux, and E. Thomé. *A heuristic quasi-polynomial algorithm for discrete logarithm in finite fields of small characteristic*, EUROCRYPT 2014.

- Soon after, Granger, Kleinjung, Zumbrägel [GKZ18] proposed another one, with a promise: **getting closer to a rigorous algorithm**

[GKZ18] R. Granger, T. Kleinjung, and J. Zumbrägel. *On the discrete logarithm problem in finite fields of fixed characteristic*, Transactions of the American Mathematical Society, 2018.

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- Suitable representation? Field $\mathbb{F}_{q^d}[x]/(J)$ where J is an irreducible polynomial in $\mathbb{F}_{q^d}[x]$ such that

$$x^q \equiv h_0/h_1 \bmod J$$

with h_0 and h_1 polynomials in $\mathbb{F}_{q^d}[x]$ of degree at most 2

- Expected time $q^{\log_2(\deg(J)) + O(d)}$

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- **Descent:** Given any polynomial Q in $\mathbb{F}_{q^d}[x]$ find integers e_f, for f in $\mathfrak{F}$, such that

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- To solve the DLP, it is sufficient to have an efficient descent

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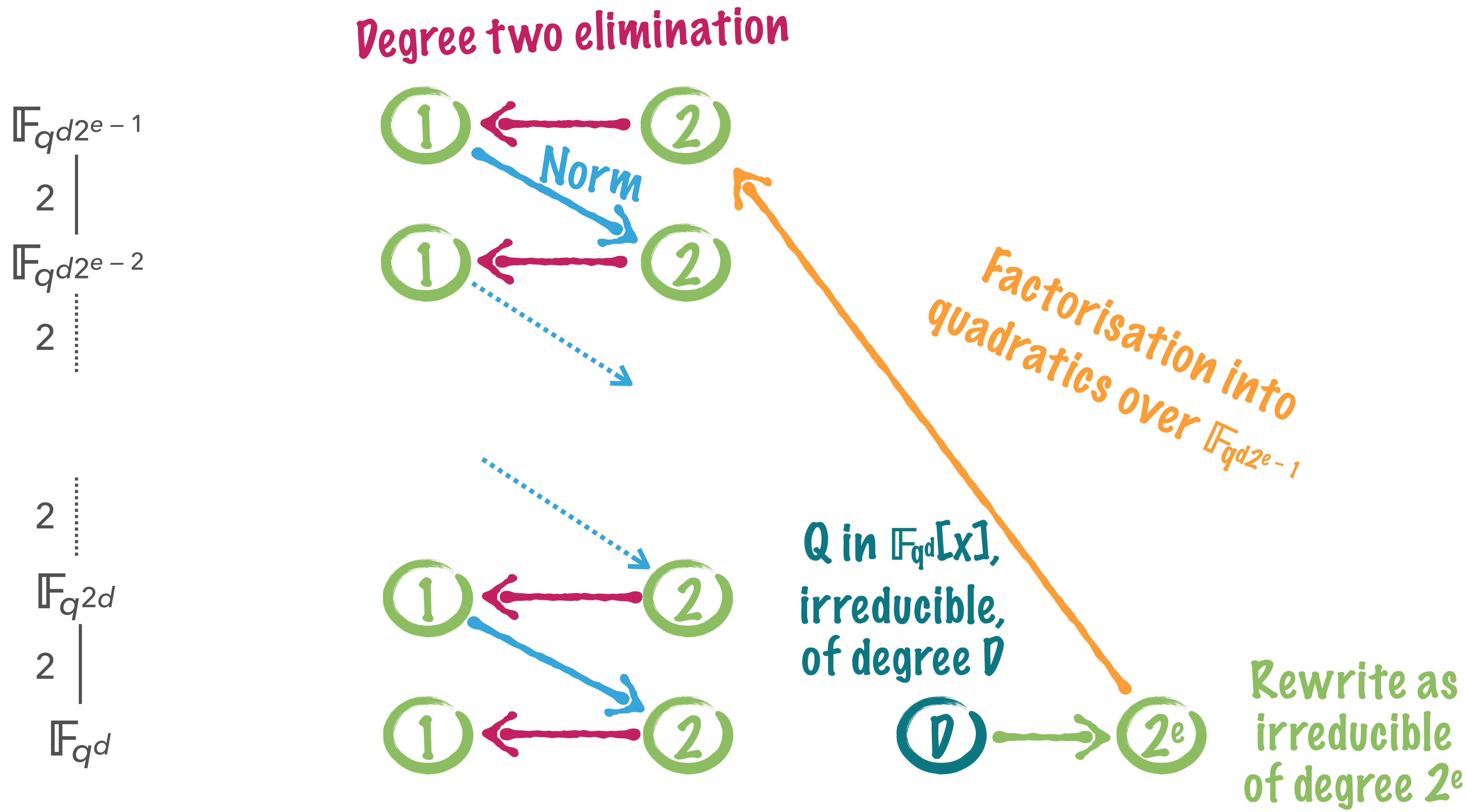
- Find **linear** polynomials $L_1, L_2, \ldots, L_m$ in $k[x]$ such that

$$Q \equiv \prod_{i=1}^{m} L_i \mod J.$$

ZIGZAG DESCENT

The **zigzag descent**: transform the degree two elimination into a **full descent algorithm**



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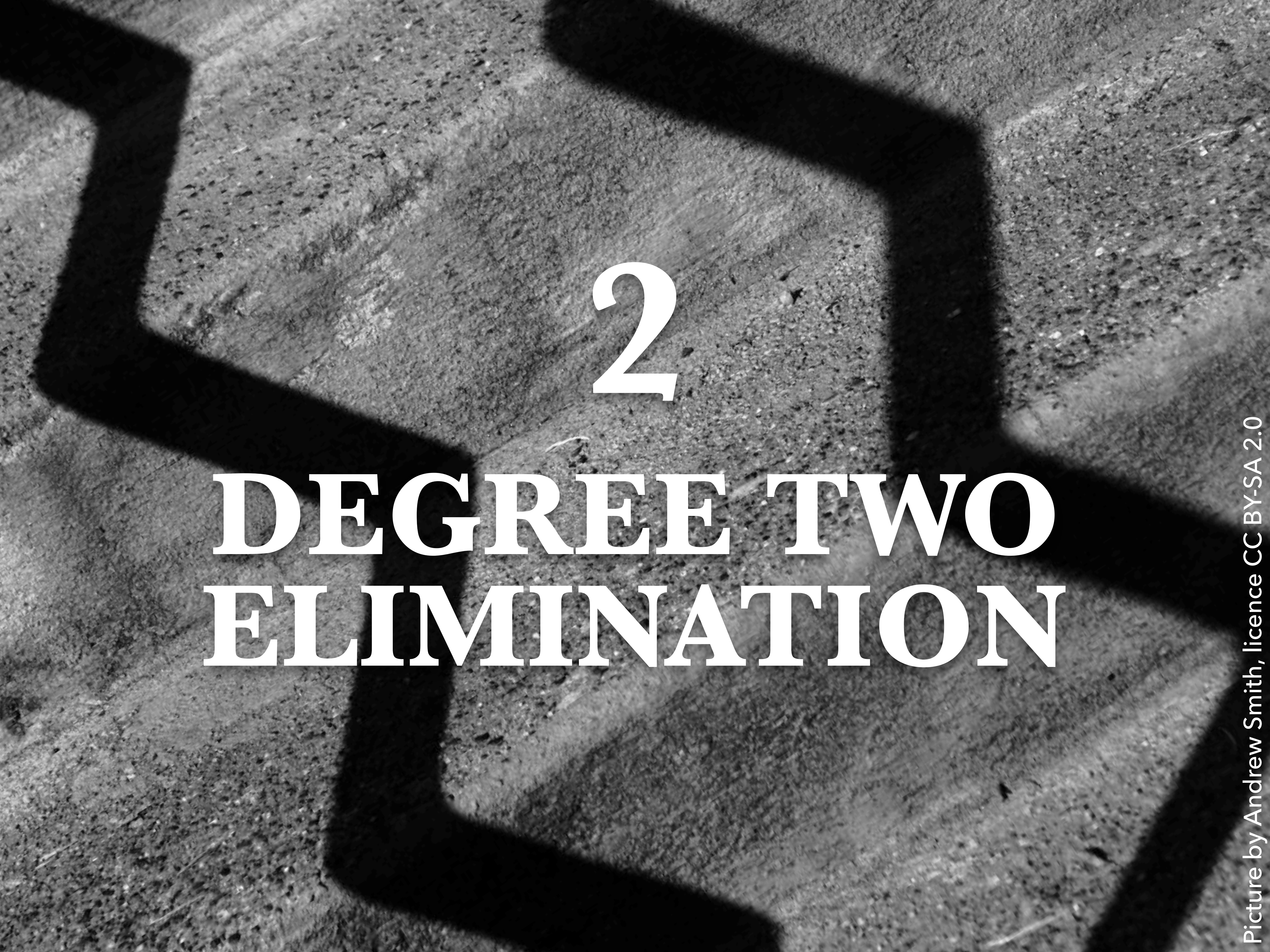
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DEGREE TWO
ELIMINATION
Picture by Andrew Smith, licence CC BY-SA 2.0

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- Key idea (from [GGMZ13]): polynomials of the form

$$\alpha x^{q+1} + \beta x^q + \gamma x + \delta \qquad \text{in } k[x]$$

  have a high probability to split over k (around q^{-3})

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- Let V be the vector space of dimension 4 of these polynomials, i.e., $V = \mathrm{span}(x^{q+1}, x^q, x, 1) \subset k[x]$

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- We have $x^q \equiv h_0/h_1 \bmod J$, so

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- Consider the vector subspace V_Q of dimension 2 in V, where Q divides the right-hand side:

$$V_Q = \{\alpha x^{q+1} + \beta x^q + \gamma x + \delta \mid \alpha x h_0 + \beta h_0 + \gamma x h_1 + \delta h_1 \equiv 0 \bmod Q\}$$

THE DEGREE TWO ELIMINATION

- For any $f = ax^{q+1} + \beta x^q + \gamma x + \delta$ in V_Q,

$$h_1 f \equiv axh_0 + \beta h_0 + \gamma x h_1 + \delta h_1 \bmod J$$

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- If f splits into linears $L_1, \ldots, L_{q+1}$ in $k[x]$, then

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- **Algorithm**: choose random polynomials f in V_Q until it splits over k. Equivalently, sample f from the projective line $\mathbb{P}(V_Q)$.

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- Then, by the absolute irreducibility, $C(k)$ has a lot of points, therefore a lot of polynomials in $\mathbb{P}(V_Q)$ split over k

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THE ACTION OF PGL_2 ON $X^q - X$

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Given $f \in k[x]$ and a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in GL_2, we define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * f(x) = (cx + d)^{\deg f} f\left(\frac{ax + b}{cx + d}\right)$$

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Lemma: $\mathbb{P}(V) \setminus S = PGL_2 * (x^q - x)$.

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Recall that we want to construct a curve C defined over k, and a surjective morphism $\theta : C \to \mathbb{P}(V_Q)$ such that

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$$C = \{(u, r_1, r_2, r_3) \mid r_1, r_2, r_3 \text{ are three dist. roots of } u\}$$

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X_1 = \{(u, r_1)\} \cong \mathbb{P}^1 & \subset \mathbb{P}(V_Q) \times \mathbb{P}^1 & \\
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- For the irreducibility of X_2: observe that $X_2 = X_1 \times_{X_0} X_1 \setminus \Delta$, and deduce the irreducibility from the ramification properties of $\theta_1 : X_1 \to X_0$

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Fibre product

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Fibre product

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Fibre product → $\times_{X_0}$; Diagonal → Δ

- For the irreducibility of X_2: observe that $X_2 = X_1 \times_{X_0} X_1 \setminus \Delta$, and deduce the irreducibility from the ramification properties of $\theta_1 : X_1 \to X_0$
- For X_3, same idea with $X_3 = X_2 \times_{X_1} X_2 \setminus \Delta$

THORSTEN KLEINJUNG*

BENJAMIN WESOLOWSKI*

A NEW PERSPECTIVE ON THE POWERS OF TWO DESCENT

for discrete logarithms in finite fields

PRESENTED AT ANTS-XIII, MADISON, WI, USA, ON THE 20/07/2018 BY BENJAMIN WESOLOWSKI

*EPFL, LAUSANNE, SWITZERLAND

COUNTING SPLIT POLYNOMIALS

- We have a cover $\theta : C \to \mathbb{P}(V_Q)$ defined over k such that

➡ C is absolutely irreducible

➡ For any P in $C(k)$, the poly. $\theta(P)$ splits completely over k

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  - ➡ θ is "$(q^3 - q)$-to-one", therefore
$$|\theta(C(k))| \approx |k|/q^3 \approx |\mathbb{P}(V_Q)(k)|/q^3$$