BENJAMIN
WESOLOWSKI

GENERATING SUBGROUPS OF RAY CLASS GROUPS with small prime ideals

AT ANTS-XIII, MADISON, WI,USA, ON THE 19/07/2018 BY BENJAMIN WESOLOWSKI, EPFL, LAUSANNE, SWITZERLAND

NOTE

The entire talk assumes the extended Riemann hypothesis

1 GENERATING CLASS GROUPS

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- More generally any order $\mathcal{O}$ in K has a class group $\mathrm{Cl}(\mathcal{O})$
- All these are ray class groups (or quotients thereof)
- Fix a modulus $\mathfrak{m}$ (essentially an ideal of $\mathcal{O}_K$). The **$\mathfrak{m}$-ray class group** $\mathrm{Cl}_{\mathfrak{m}}(K)$ is the quotient

$$\mathcal{I}_{\mathfrak{m}}(K) / P_{\mathfrak{m}}(K) = (\text{ideals in } \mathcal{O}_K \text{ coprime to } \mathfrak{m}) / (\text{some principal ideals})$$

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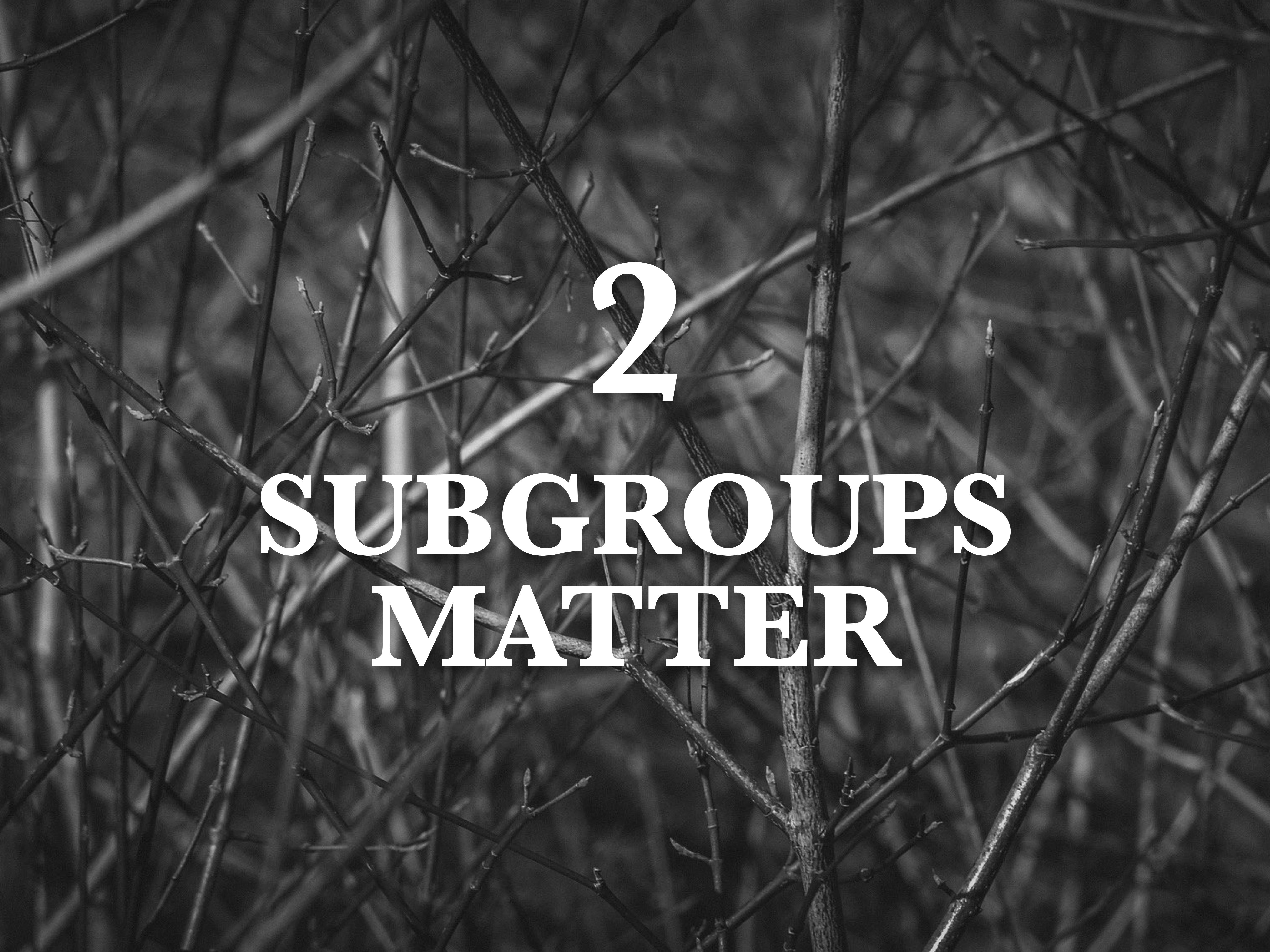
- For which bound B does S generate $\mathrm{Cl}_{\mathfrak{m}}(K)$?
- Answer from [Bach90]: $B = 18 \log(\mathrm{Disc}(K)^2 N(\mathfrak{m}))^2$ works!

[Bach90] Eric Bach. *Explicit bounds for primality testing and related problems*, Mathematics of Computation, 1990.

GENERATING SUBGROUPS OF RAY CLASS GROUPS

- What if we want generators of a **subgroup** of $\mathrm{Cl}_{\mathfrak{m}}(K)$, are small prime ideals still sufficient?



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SUBGROUPS MATTER

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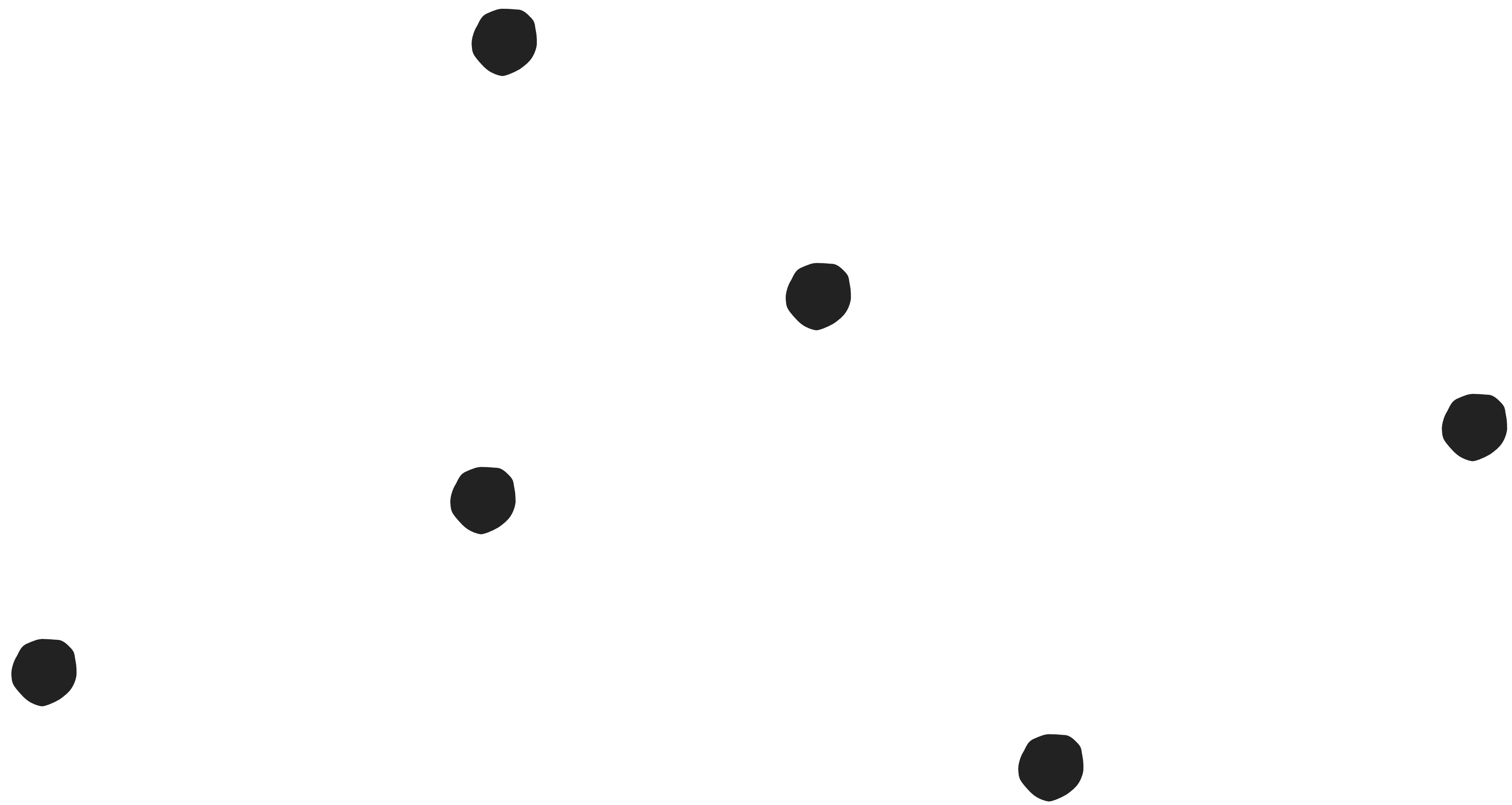
- Bounds on degrees of computable isogenies to get connected isogeny graphs
- An algorithm to find short vectors in cyclotomic ideal lattices

WHAT IS AN ISOGENY GRAPH?



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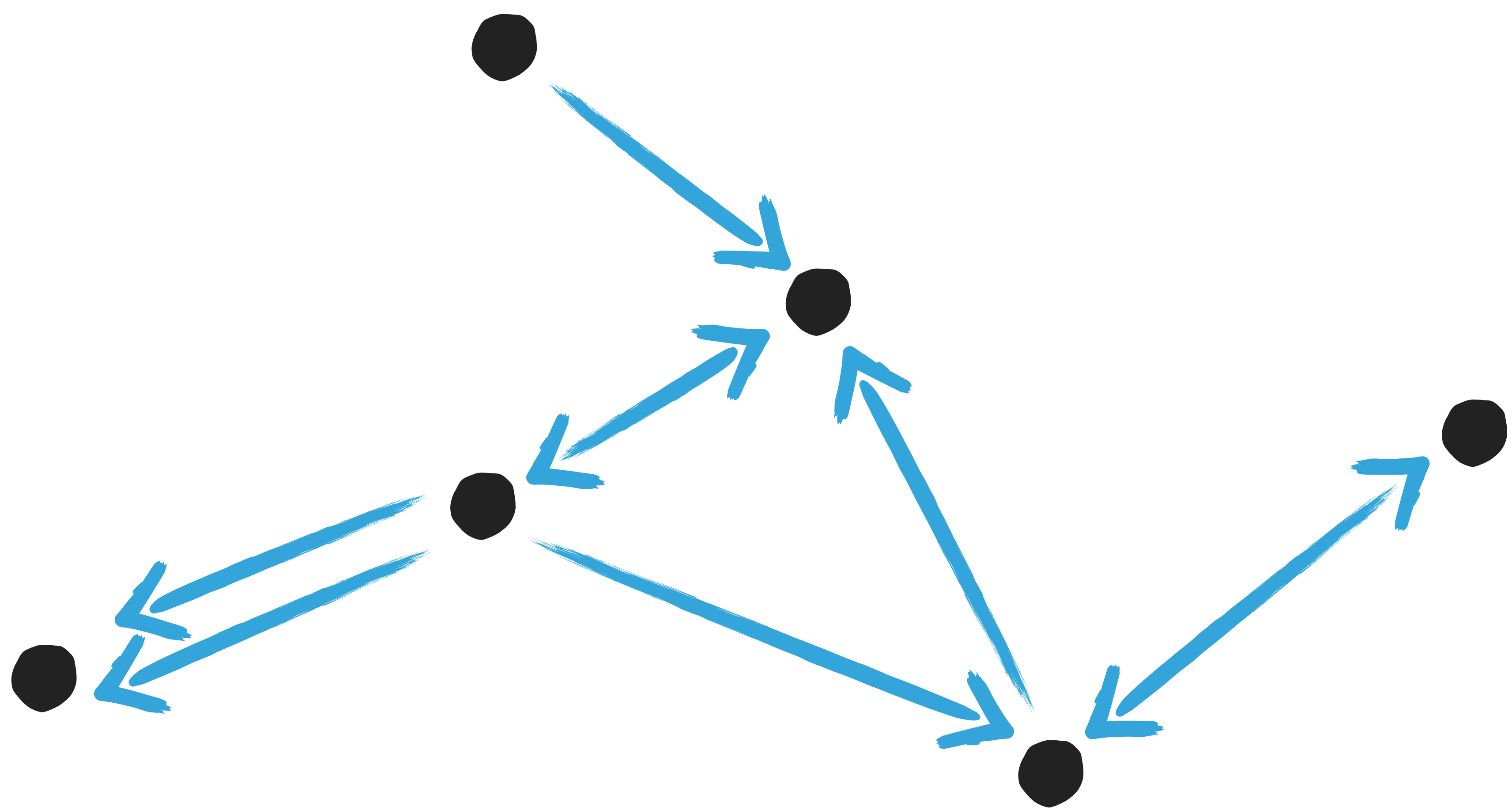

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Edges represent isogenies between them

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- Let Hor(A) be the set of abelian varieties isogenous to A with same endomorphism ring
- For any invertible ideal $\mathfrak{l}$ in End(A), the isogeny $A \longrightarrow A/A[\mathfrak{l}]$ is horizontal, of degree $N(\mathfrak{l})$

CONNECTED ISOGENY GRAPHS

Graph with vertices $\mathrm{Hor}(A)$, and edges isogenies of prime norm at most B	$\cong$	Cayley graph of $\mathrm{Cl}(\mathrm{End}(A))$, with generators the ideals of prime norm at most B

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If End(A) is Gorenstein

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- They correspond to a subgroup of Cl(End(A)), and the corresponding subgraph of the Cayley graph
- Bound B on the degree of isogenies to get a connected graph where the isogenies can be computed?

ISOGENY GRAPHS TO STUDY THE DLP

Isogeny graphs are a central tool for studying the DLP

- Galbraith, Hess, and Smart, *Extending the GHS Weil descent attack*, EUROCRYPT 2002
- Jao, Miller, and Venkatesan, *Do All Elliptic Curves of the Same Order Have the Same Difficulty of Discrete Log?*, ASIACRYPT 2005
- Smith, *Isogenies and the Discrete Logarithm Problem in Jacobians of Genus 3 Hyperelliptic Curves*, EUROCRYPT 2008
- Jetchev and Wesolowski, *Horizontal isogeny graphs of ordinary abelian varieties and the discrete logarithm problem* (preprint)

SVP IN CYCLOTOMIC IDEAL LATTICES

- A typical choice of lattices for lattice-based cryptographic schemes: ideals in a cyclotomic field K (seen as lattices via the Minkowski embedding)

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[CDW17] R. Cramer, L. Ducas, and B. Wesolowski. *Short Stickelberger class relations and applications to Ideal-SVP*, EUROCRYPT 2017.

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= the relative class group, a subgroup of Cl(K)

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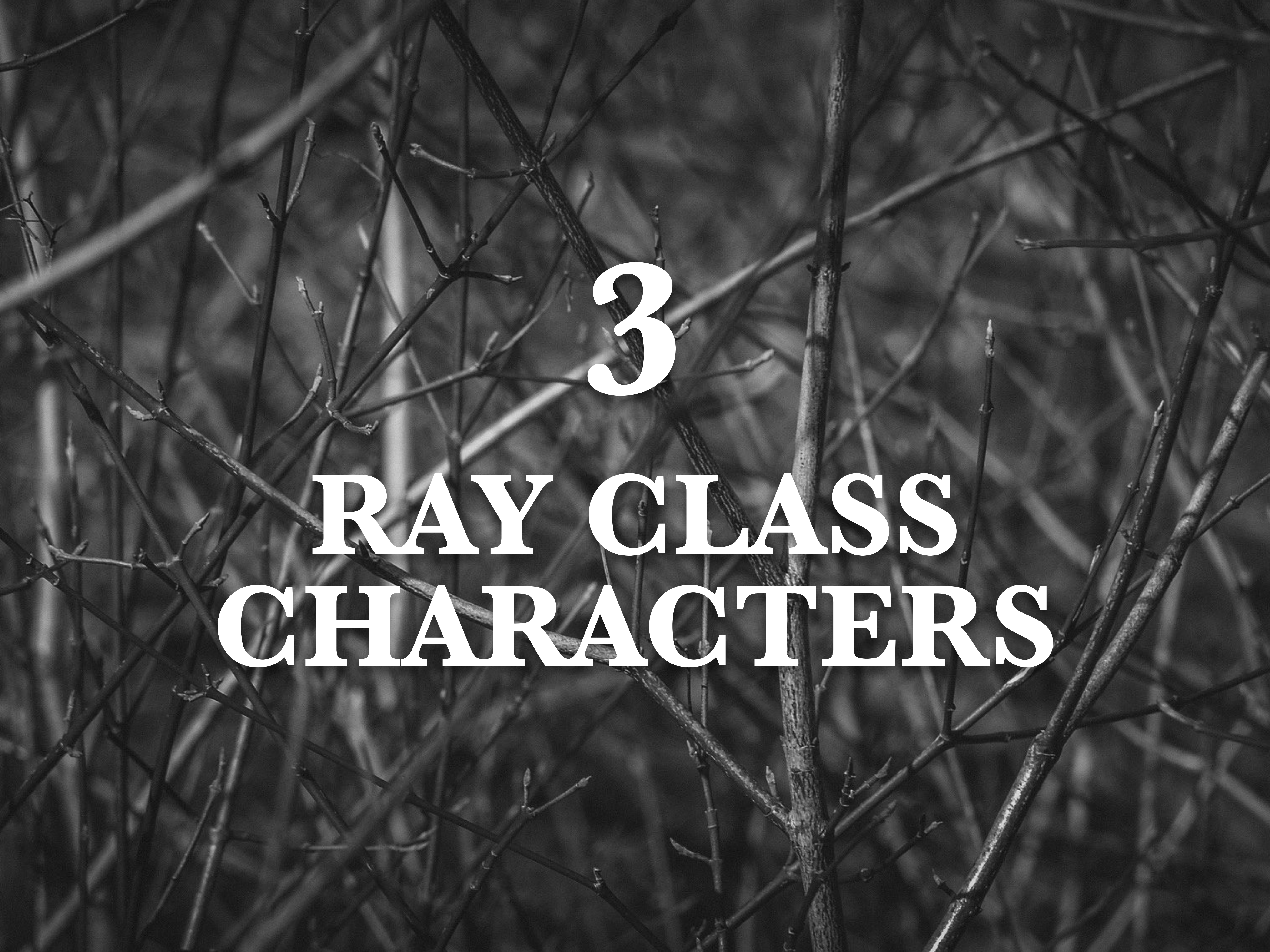

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RAY CLASS CHARACTERS

MAIN THEOREM

Theorem: Let H be any subgroup of $\mathrm{Cl}_{\mathfrak{m}}(K)$, and consider a character $\chi\colon \mathrm{Cl}_{\mathfrak{m}}(K) \longrightarrow \mathbb{C}^\times$ that is not trivial on H. Then, there is an ideal $\mathfrak{p}$ such that

- $N(\mathfrak{p})$ is prime,
- $(\mathfrak{p}, \mathfrak{m}) = 1$,
- $[\mathfrak{p}]_{\mathfrak{m}} \in H$,
- $\chi(\mathfrak{p}) \neq 1$,
- $N(\mathfrak{p}) \leq ([\mathrm{Cl}_{\mathfrak{m}}(K) : H]\,(2.71 \log(\mathrm{Disc}(K)\, N(\mathfrak{m})) + 1.29\, |\mathfrak{m}_\infty|$
  $+ 1.38\, \omega(\mathfrak{m})) + 4.13)^2$

This bound is $O\big([\mathrm{Cl}_{\mathfrak{m}}(K) : H]^2 \log(\mathrm{Disc}(K)\, N(\mathfrak{m}))^2 \big)$

MAIN THEOREM

- Proof uses analytic methods similar to [Bach90]
- Play with characters of $\mathrm{Cl}_{\mathfrak{m}}(K)/H$ to account for the extra condition that the ideals to consider are in the subgroup H

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- Let S be the set of all ideals $\mathfrak{p}$ of prime norm smaller than the bound B from the theorem, and such that $[\mathfrak{p}]_{\mathfrak{m}} \in H$

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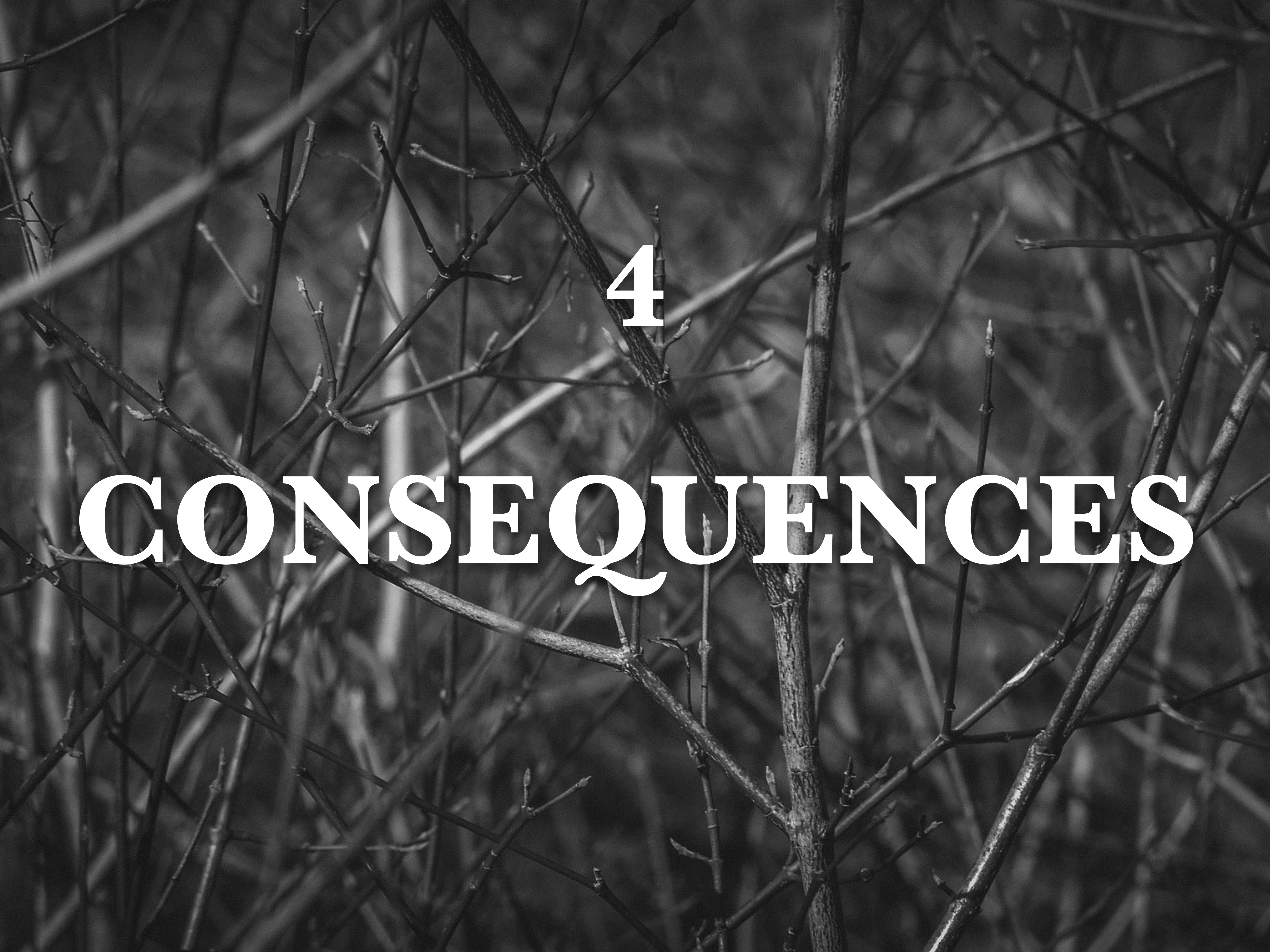
- This character extends to a character $\chi\colon \mathrm{Cl}_{\mathfrak{m}}(K) \longrightarrow \mathbb{C}^{\times}$ that is not trivial on H
- From the theorem, there is an ideal $\mathfrak{p}$ in S such that $\chi(\mathfrak{p}) \neq 1$
- So $\mathfrak{p}$ is in N and $\chi(\mathfrak{p}) \neq 1$, a contradiction, so $N = H$



4 CONSEQUENCES

ON INTEGERS

- Let m be a positive integer, and H a subgroup of $(\mathbb{Z}/m\mathbb{Z})^\times$

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- Let m be a positive integer, and H a subgroup of $(\mathbb{Z}/m\mathbb{Z})^\times$
- H is generated by the prime numbers p such that $p \bmod m$ is in H and $p \leq 16\,([(\mathbb{Z}/m\mathbb{Z})^\times : H] \log m)^2$

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- $G\left(26(h^+ \log(\mathrm{Disc}(K)N(\mathfrak{f})))^2\right)$ **is connected**, where h^+ is the narrow class number of the real suborder of $\mathrm{End}(A)$

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PROOF OF THE MAIN THEOREM

- For any $0 < a < 1$, $x > 0$ and ideal $\mathfrak{a}$, let

$$P(\mathfrak{a}, x) = \Lambda(\mathfrak{a}) \left(\frac{N(\mathfrak{a})}{x} \right)^a \log\left(\frac{x}{N(\mathfrak{a})} \right)$$

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Lemma [Bach90]: For any character η,

$$\sum_{N(\mathfrak{a}) < x} \eta(\mathfrak{a}) P(\mathfrak{a}, x) = \frac{-1}{2\pi i} \int_{2 - i\infty}^{2 + i\infty} \frac{x^s}{(s + a)^2} \frac{L'_\eta}{L_\eta}(s)\, ds$$

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Logarithmic derivative of the Hecke L-function associated to η

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- The right-hand side is estimated as $x + O(x^{1/2})$
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- So such an x cannot be too large

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- **Proof of our new bounds:** similar ideas, but play with characters of $\mathrm{Cl}_{\mathfrak{m}}(K)/H$ to account for the extra condition that the ideals $\mathfrak{a}$ are in the subgroup H