Computing A Database of Belyĭ Maps

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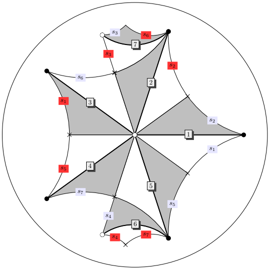
Joint work with Michael Musty, Jeroen Sijsling, and John Voight.



July 16, 2018

Outline

(I) Definitions and motivation

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(II) Overview of method

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(III) Progress on computations

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(I) Definitions and motivation
(II) Overview of method
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(V) Analysis of the data

Definitions and motivation

What is a Belyĭ map?

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A *Belyĭ map* over $\mathbb{C}$ is a nonconstant morphism of algebraic curves $\phi : X \to \mathbb{P}^1_{\mathbb{C}}$ that is unramified outside $\{0, 1, \infty\}$.

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$$\phi(x) = 2x^3 + 3x^2 = x^2(2x+3)$$
$$\phi(x) - 1 = 2x^3 + 3x^2 - 1 = (2x-1)(x+1)^2.$$

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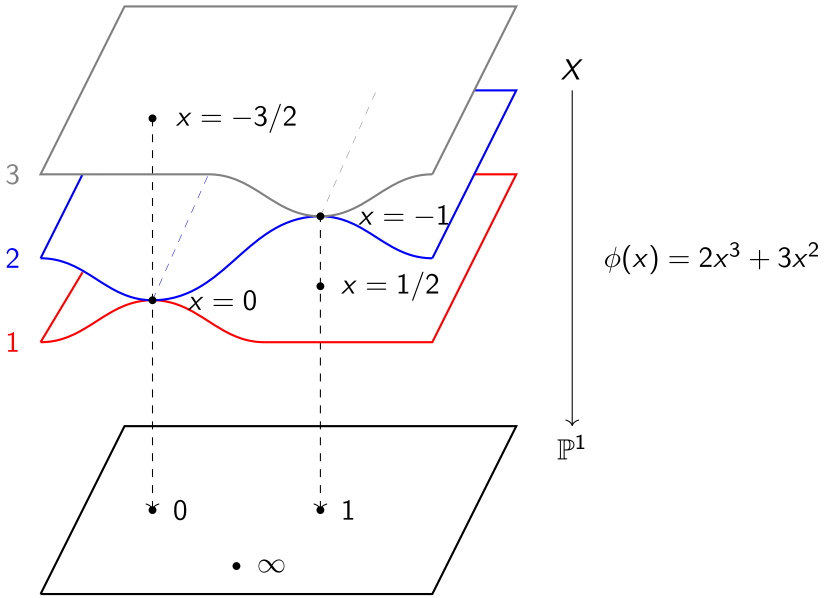

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Goal: Understand $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by studying this action.

We compute explicit equations for Belyĭ maps in order to get a concrete view of this action.

'A. Grothendieck and his students developed a combinatorial description ("maps") of finite coverings [of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$]... It has not aided in understanding the Galois action. We have only a few examples of non-solvable coverings whose Galois conjugates have been computed.'

—Pierre Deligne

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- *permutation triples*
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- *dessins d'enfants*

Permutation Triples

A *transitive permutation triple* of degree d is a triple $\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$ such that

- $\sigma_\infty \sigma_1 \sigma_0 = 1$, and
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Two permutation triples σ, σ' are *simultaneously conjugate* if there exists $\rho \in S_d$ such that

$$(\sigma_0', \sigma_1', \sigma_\infty') = (\rho\sigma_0\rho^{-1}, \rho\sigma_1\rho^{-1}, \rho\sigma_\infty\rho^{-1}) .$$

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The *size* of a passport is the number of permutation triples belonging to it, up to simultaneous conjugacy.

Passports

Consider the permutation triple $\sigma = (\sigma_0, \sigma_1, \sigma_\infty)$ where

$$\sigma_0 = (1\,3\,7)(2)(4\,5\,6)\,,\ \ \sigma_1 = (1\,4\,5\,3)(2\,7)(6)\,,\ \ \sigma_\infty = (1\,2\,7\,5)(3)(4\,6)\,.$$

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One can show that this passport has size 2; the other triple is $\sigma' = (\sigma'_0, \sigma'_1, \sigma'_\infty)$ with

$$\sigma'_0 = (1\,3\,7)(2)(4\,5\,6)\,, \;\; \sigma'_1 = (1\,6\,3\,2)(4\,5)(7)\,, \;\; \sigma'_\infty = (1\,7\,6\,4)(2\,3)(5)\,.$$

Triangle Groups

For a triple of integers $a, b, c \in \mathbb{Z}_{\geq 2}$, we define the *triangle group*

$$\Delta(a, b, c) = \langle \delta_a, \delta_b, \delta_c \mid \delta_a^a = \delta_b^b = \delta_c^c = \delta_c \delta_b \delta_a = 1 \rangle .$$

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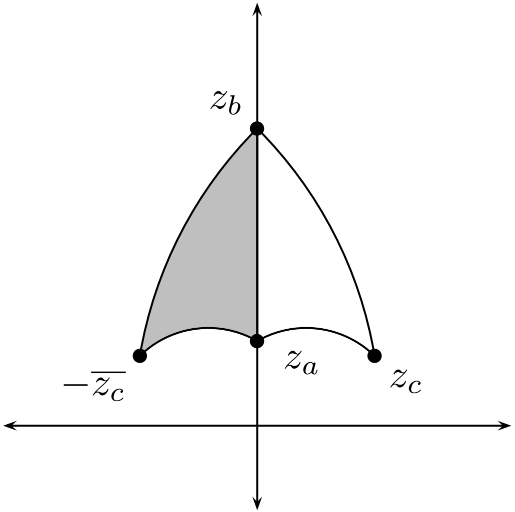

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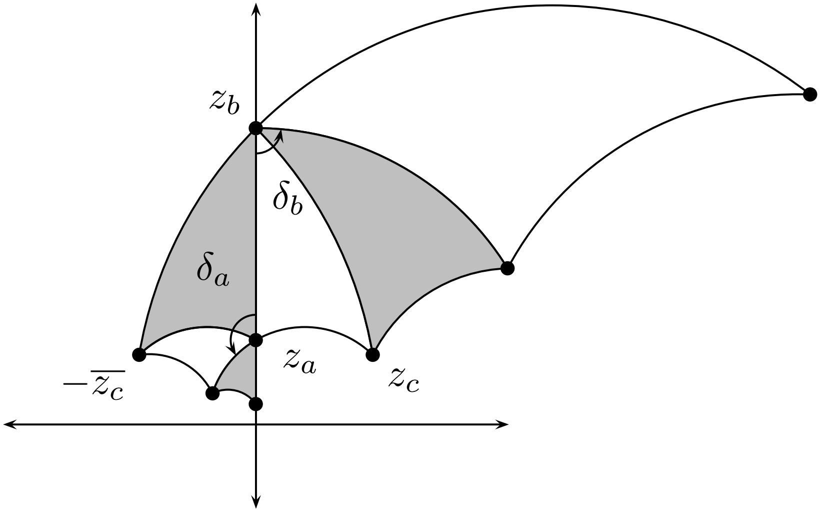

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Galois action for example `7T5-[3,4,4]-331-421-421-g0`

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$$\phi - 1 = \lambda \cdot \frac{\left(x-\frac{1}{189}\left(44\sqrt{7}+140\right)\right)^4\left(x-\frac{1}{7}\left(12\sqrt{7}-28\right)\right)^2\left(x-\frac{1}{14}\left(3\sqrt{7}+7\right)\right)^1}{\left(x-\frac{4}{21}\left(\sqrt{7}+3\right)\right)^2\left(x-\frac{4}{21}\left(\sqrt{7}+1\right)\right)^4}$$

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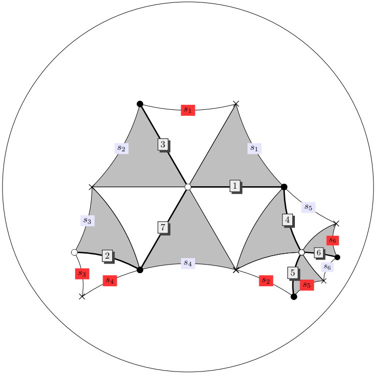

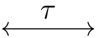

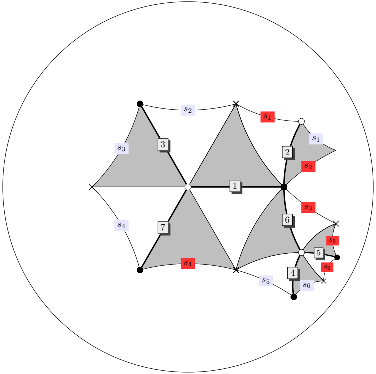

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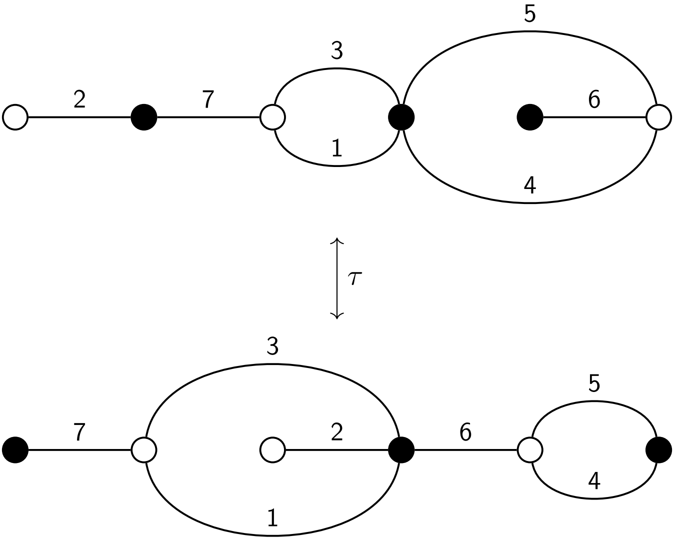

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5. Verify that ϕ has the correct ramification and monodromy.

Progress on computations

Completeness of computation

d \ g	0	1	2	3	≥ 4	total
1	1/1	0	0	0	0	1/1
2	1/1	0	0	0	0	1/1
3	2/2	1/1	0	0	0	3/3
4	6/6	2/2	0	0	0	8/8
5	12/12	6/6	2/2	0	0	20/20
6	38/38	29/29	7/7	0	0	74/74
7	89/89	50/50	7/13	2/3	0	148/155
8	81/261	83/217	0/84	0/11	0	164/573
9	97/583	33/427	0/163	0/28	0/6	130/1207

Newton's Method

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In this paper we describe how to use Newton's method to compute genus 1 Belyĭ maps. This has allowed us to extend our computations to higher degrees.

Example

Consider the permutation triple $\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in {S_6}^3$ with

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Then σ belongs to the passport `6T15-[5,5,5]-51-51-51-g1`.

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From the Riemann-Hurwitz formula, we see that this corresponds to a Belyĭ map ϕ defined on a genus 1 curve X. Let K be the pointed field of definition of ϕ, so X has a K-rational point. Then X can be written in Weierstrass form:

$$X : y^2 = x^3 - 27c_4x - 54c_6$$

for some $c_4, c_6 \in K$.

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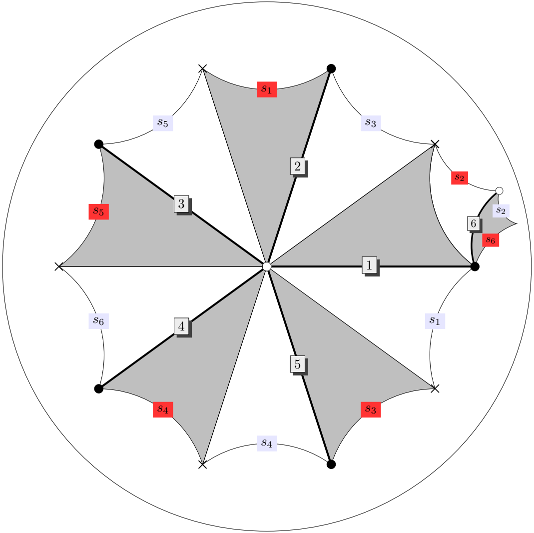

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Since $\{1, x\}$ and $\{1, x, y, x^2, xy, x^3, x^2y\}$ are bases for $\mathcal{L}(2\infty)$ and $\mathcal{L}(7\infty)$, respectively, then

$$\phi = \frac{\phi_0}{\phi_\infty} = u\frac{a_0 + x}{b_0 + b_2x + b_3y + \cdots + b_6x^3 + x^2y}$$

for some $u, a_0, b_0, \ldots, b_6 \in K$.

Example

To determine the coefficients c_4, c_6 of the curve and $u, a_0, b_0, \ldots, b_6$ of the map, we first use our numerical method to obtain a rough approximation. We then apply Newton's method to the system of equations obtained as follows.

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Since σ_1 has cycle type $5, 1$, then there are two points $P_{1,b}, P_{2,b}$ above 1. Since X is smooth, then the complete local ring $\widehat{\mathbb{C}[X]}_{P_{i,b}}$ is a discrete valuation ring.

Example

To determine the coefficients c_4, c_6 of the curve and $u, a_0, b_0, \ldots, b_6$ of the map, we first use our numerical method to obtain a rough approximation. We then apply Newton's method to the system of equations obtained as follows.

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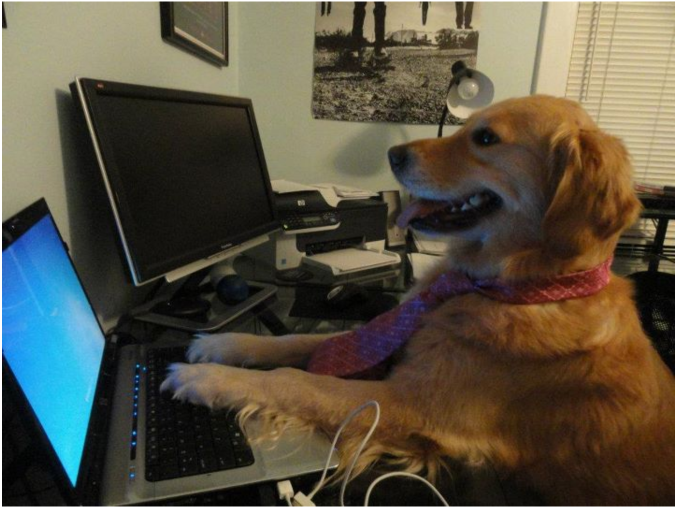
Expressing x, y in terms of the uniformizer for this local ring, the condition that $\phi - 1$ has zeroes of order 5 (resp., 1) at $P_{1,b}$ (resp., $P_{2,b}$) imposes equations on the coefficients.

Live demo



Example

The passport containing σ has size 8. We find that it splits into 4 Galois orbits, each of size 2:

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$$\{((1,2,3,4,5),(1,3,5,6,2),(1,3,6,5,4)),\ ((1,2,3,4,5),(1,3,5,4,6),(1,3,2,6,4))\}$$

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$$\{((1,2,3,4,5),(1,4,6,5,2),(1,6,4,3,5)),\ ((1,2,3,4,5),(2,3,5,4,6),(1,3,6,4,2))\}$$

Analysis of the data

Reducibility of passports

One expects a “generic” passport to be irreducible. So what is the reason behind this splitting?

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We leave to future work explanations for each instance of a reducible passport in the database.

Reducibility of passports

Suppose $\mathcal{P}$ is a passport of size ℓ that decomposes into Galois orbits of sizes $\ell_1, \ldots, \ell_r$.

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To measure the irreducibility of $\mathcal{P}$, define the weight

$$\mathrm{w}(\mathcal{P}) := \begin{cases} 1, & \text{if } r = 1; \\ \frac{1}{(\ell-1)^2} \sum_{i=1}^{r} (\ell_i - 1)^2, & \text{if } r \geq 2. \end{cases}$$

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For example, the passport `6T15-[5,5,5]-51-51-51-g1` above (of size $8 = 2 + 2 + 2 + 2$) has weight

$$\frac{1}{(8-1)^2}((2-1)+(2-1)+(2-1)+(2-1)) = \frac{4}{49}\,.$$

Reducibility of passports

Let $\mathcal{P}_d$ be the set of passports of degree at most d and define the reducibility constant

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From the database we find that $\beta(d) = 1$ for $d \leq 4$, $\beta(5) \approx 0.9393$, $\beta(6) \approx 0.9444$, and $0.8779 < \beta(7) < 0.9046$.

Access to data

`beta.lmfdb.org/Belyi/`

Thank you!