Computing Hecke eigenvalues analytically

Nathan C. Ryan
(joint with Alex Ghitza, Darío Terán, David Armendáriz and Owen Colman)

Thirteenth Algorithmic Number Theory Symposium
July 16, 2018

Conjectures related to Hecke eigenvalues:

- Classical modular forms:
  1. Lehmer's conjecture
  2. Sato-Tate conjecture
- Siegel modular forms:
  1. Paramodular conjecture
  2. Harder's conjecture
  3. Existence of lifts
  4. Some pathological forms (rational nonlift eigenforms in 2-dimensional spaces)

Crash course on Siegel MFs

- The group is $\Gamma^{(2)} = \mathrm{Sp}(4)$ (instead of $\Gamma^{(1)} = \mathrm{SL}(2)$)
- The upper half space $\mathcal{H}_2$ is

$$\{Z = X + iY :\in M_{2\times 2}(\mathbb{C}) : Y > 0\}$$

(instead of $\mathcal{H}_1 = \{z \in \mathbb{C} : \Im(z) > 0\}$).
- The Hecke algebra is generated by T_p and T_{p^2} (instead of just T_p).
- The slash operator is, for $\alpha = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}(4)$

$$(F \mid_k \alpha)(Z) = \det(CZ + D)^{-k} F((AZ + B)(CZ + D)^{-1}).$$

- The Fourier expansion of F is of the form

$$\sum_{T \geq 0, {}^t T = T} a(T) e^{2\pi i \mathrm{tr}(TZ)}.$$

How do we find Hecke eigenvalues?

- Classical modular forms:
  - Modular symbols method: in principle works for all N and k; in practice (in Sage) it slows down around

$$(N, k) = 2000, 2; \quad (N, k) = (1, 200); \quad (N, k) = (50, 50).$$

- Siegel modular forms:
  - Only real way (up until very recently) has been to compute the expansion of the eigenform F, compute the action of T_p and T_{p^2} on F (quite ad hoc!).
  - Has only been done systematically for levels ≤ 4.
  - To compute T_{p^2} for $p \approx 10^2$ we need coefficients indexed by quadratic forms of discriminant up to $\approx 3 \cdot 10^6$.

Yikes!

Our proposal (similar to what Hejhal and Venkatesh et al do for Maass forms):

- For classical modular form f, an eigenform:
  - Fix $z \in \mathcal{H}_1$, evaluate $f(z)$, $(f \mid_k T_p)(z)$ and the ratio

$$\lambda_p = \frac{(f \mid_k T_p)(z)}{f(z)}$$

- For Siegel modular form F, an eigenform:
  - Fix $Z \in \mathcal{H}_2$, evaluate $F(Z)$, $(F \mid_k T_p)(Z)$, $(F \mid_k T_{p^2})(Z)$ and the ratios

$$\lambda_p = \frac{(F \mid_k T_p)(Z)}{F(Z)} \text{ and } \lambda_{p^2} = \frac{(F \mid_k T_{p^2})(Z)}{F(Z)}$$

Conceptual shift:

> *Instead of representing a modular form as a list of Fourier coefficients, we can represent it as a set of points in $\mathcal{H}$ and the values it takes on at those points.*

What do we gain?

level	weight	p	modular symbols	analytic
1	12	1 000	**0.109**	0.166
		10 000	**1.270**	1.350
		100 000	14.500	**13.000**
		1 000 000	177.000	**136.000**
1	24	1 000	**0.300**	0.329
		10 000	2.510	**1.690**
		100 000	30.900	**16.800**
		1 000 000	351.000	**178.000**
1	100	1 000	1.350	**1.180**
		10 000	15.600	**4.670**
		100 000	207.000	**53.400**
1	200	1 000	**2.698**	3.052
		10 000	36.120	**26.634**
2	8	1 000	**0.053**	0.138
		10 000	**0.718**	1.391
		100 000	**9.233**	15.596
		1 000 000	**105.630**	148.558
2	48	1 000	0.397	**0.280**
		10 000	5.398	**2.954**
		100 000	71.526	**34.375**
3	6	1 000	**0.047**	0.141
		10 000	**0.567**	1.426
		100 000	**7.253**	15.772
		1 000 000	**82.216**	161.781

What do we gain?

f	numerical (s)	standard (s)
Υ_{20}	57	240
Υ_{22}	59	410
$\Upsilon_{24\mathrm{a}}$	59	559
$\Upsilon_{24\mathrm{b}}$	59	563
$\Upsilon_{26\mathrm{a}}$	59	658
$\Upsilon_{26\mathrm{b}}$	60	659

And we think we can get even faster!

What do we lose?

- Numerical values for the eigenvalues instead of exact values
  - Not that big a deal: since we know the field we can use LLL to find an exact expression for the numbers we've computed numerically and because often our next step is to compute associated L-functions and those want numerical values for the eigenvalues any way
- Already for classical forms of small level > 4, the code slows down considerably
  - might be able to address this by computing expansions at other cusps

How is it done?

Classical Modular Forms:

1. Prove a theorem that lets you control the size of the tail:

Proposition

Let $\varepsilon > 0$ and $y = \Im(z)$. If T is such that

$$T \geq \frac{d}{2\pi y} \qquad \text{and} \qquad \frac{d+1}{2\pi y} e^{-2\pi y T} T^d < \varepsilon,$$

then $|f(z) - f_T(z)| < \varepsilon$.

2. When you evaluate $(f \mid_k T_{10007})(z)$ you have to calculate something like
$$f\left(\frac{z+17}{10007}\right).$$
For such an element of $\mathcal{H}_1$, the Fourier expansion converges slowly. So apply Fricke.

3. Use symmetry when the real part of z is 0 to cut the number of evalutions in half:

Corollary

Given f as in the Proposition:

(a) $f(iy) \in \mathbb{R}$ for all $y \in \mathbb{R}_{>0}$;

(b) if $y \in \mathbb{R}_{>0}$, p is prime and $b \in \{1, \ldots, p-1\}$ then

$$f\left(\frac{iy-b}{p}\right) = \overline{f\left(\frac{iy+b}{p}\right)}.$$

(c) Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$. The summands in $T_p f(iy)$ (for $p > 2$) come in pairs of conjugate complex numbers, which allows us to reduce the necessary computation in half. In particular,

$$(T_p f)(iy) = p^{k-1} f(iyp) + \frac{1}{p} f\left(i\frac{y}{p}\right) + \frac{2}{p}\left(\mathrm{Re} f\left(\frac{iy+1}{p}\right) + \cdots + \mathrm{Re} f\left(\frac{iy+\frac{p-1}{2}}{p}\right)\right).$$

Siegel vs. Classical

- Siegel modular form coefficients are indexed by positive semi-definite quadratic forms.
- Need to calculate λ_p and λ_{p^2}.
- No Deligne bound on the coefficients and so we cannot control the error that well.
- The fundamental domain is bounded by 28 algebraic surfaces and so we stick with level 1 for now.
- The ring of level 1 Siegel modular forms is generated by E_4, E_6, χ_{10} and χ_{12}.
- To calculate λ_p in the standard way, we need coefficients of F up to discrimint p^3. To calculate it analytically using the best known bound on coefficients we need roughly the same number of coefficients. So we work differently. . . .

Instead...

1. we express a Siegel modular form F as a polynomial in the generators E_4, E_6, χ_{10} and χ_{12}.
2. using work of Lauter and Bröker, we evaluate the generators at our choice of Z. This is easy to do because the coefficients of the generators are easy to compute.
3. we evaluate F by substituting the values we found in Step 2 into the polynomial we found in Step 1.

In particular, we compute the eigenvalues of F by computing almost no coefficients of F.

Next steps

1. Optimize the calculation of coefficients of the generators
2. Find some kind of symmetry in the Siegel setting
3. Try to understand the nonlift eigenforms in weight 24 and 26 that are rational but live in a two dimensional space.